Advanced Mechatronics Engineering

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September 17, 2016

Outline

- Motivation
- Similarity Transformations
- Minimal Realizaions
- Quadratic Forms and Positive/Negative Definiteness

Consider the following linear system:

$$\dot{\mathbf{x}} = \begin{bmatrix} -56 & -15 & 30 \\ 30 & 9 & -16 \\ -90 & -25 & 48 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} u, \tag{1}$$

$$\mathbf{y} = \begin{bmatrix} -7 & -1 & 4 \end{bmatrix} \mathbf{x}. \tag{2}$$

Find the transfer function.

$$\mathbf{Q}_\mathrm{c} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} -2 & 7 & -17 \\ 1 & -3 & 7 \\ -3 & 11 & -27 \end{bmatrix}. \tag{3}$$

$|\mathbf{Q}_\mathrm{c}|$=0. Therefore, $\mathbf{Q}_\mathrm{c}^{-1}$ does not exist.

Let us try another technique

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \frac{s^2 - 3s - 4}{s^3 - s^2 - 10s - 8}, \tag{4}$$

we factorize the transfer function

$$G(s) = \frac{(s-4)(s+1)}{(s-4)(s+1)(s+2)}, \tag{5}$$

pole/zero cancellations. Therefore, $G(s)$ is give by

$$G(s) = \frac{1}{s+2}. \tag{6}$$

Another state-space representation of the system would be

$$\dot{\xi} = -2\xi + u, \tag{7}$$

$$y = -\xi. \tag{8}$$

Obviously ξ is not similar to $\mathbf{x}$. Therefore, System A, that is given by

System A

$$\dot{\mathbf{x}} = \begin{bmatrix} -56 & -15 & 30 \\ 30 & 9 & -16 \\ -90 & -25 & 48 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} u, \quad (9)$$

$$\mathbf{y} = \begin{bmatrix} -7 & -1 & 4 \end{bmatrix} \mathbf{x}. \quad (10)$$

is not similar to System B, that is given by

System B

$$\dot{\xi} = -2\xi + u, \quad (11)$$

$$y = -\xi. \quad (12)$$

System A

$$\dot{\mathbf{x}} = \begin{bmatrix} -56 & -15 & 30 \\ 30 & 9 & -16 \\ -90 & -25 & 48 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} u \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -7 & -1 & 4 \end{bmatrix} \mathbf{x}. \tag{13}$$

System B

$$\dot{\xi} = -2\xi + u \quad \text{and} \quad y = -\xi. \tag{14}$$

System B is a first-order system with respect to the input/output. However, there exist some states (internal) that can either not be reached from the input (uncontrollable states) or that cannot be seen from the output (unobservable states).

Minimal Realizations

Recipe

- Take as many columns of $\mathbf{Q}_{\rm c}$ as are linearly independent, and extend by anything that gives a full rank.

$$\widetilde{\mathbf{Q}}_{\rm c} = [\mathbf{Q}_{\rm cli}|\,\text{extension}]; \tag{15}$$

- Use $\mathbf{T} = \widetilde{\mathbf{Q}}_{\rm c}^{-1}$ for a similarity transformation, where $\mathbf{Q}_{\rm cli}$ is a matrix of the linear independent columns of $\mathbf{Q}_{\rm c}$.

$$\mathbf{Q}_c = \begin{bmatrix} -2 & 7 & -17 \\ 1 & -3 & 7 \\ -3 & 11 & -27 \end{bmatrix} \tag{16}$$

The first and second columns of $\mathbf{Q}_c$ are linearly independent. Therefore, we replace the last vector as follows:

$$\widetilde{\mathbf{Q}}_c = \begin{bmatrix} -2 & 7 & 0 \\ 1 & -3 & 0 \\ -3 & 11 & -1 \end{bmatrix} \tag{17}$$

Now, $|\widetilde{\mathbf{Q}}_c| = 1$. Use its inverse as a similarity transformation.

$$\mathbf{T} = \widetilde{\mathbf{Q}}_c^{-1} = \begin{bmatrix} 3 & 7 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}. \tag{18}$$

The similar system is given by

$$\dot{\eta} = \begin{bmatrix} 0 & -2 & 22 \\ 1 & -3 & 2 \\ 0 & 0 & 4 \end{bmatrix} \eta + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \tag{19}$$

$$\mathbf{y} = \begin{bmatrix} 1 & -2 & -4 \end{bmatrix} \eta. \tag{20}$$

We conclude that $\dot{\eta}_3 = 4\eta_3$, can not be reached from the input. Therefore, there exist an controllable mode at $\lambda = 4$. This mode is even instable.

Uncontrollable States

In general, this transformation results in

$$\dot{\eta} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ 0 & \mathbf{A}_{22} \end{bmatrix} \eta + \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix} u. \tag{21}$$

All uncontrollable eigenvalues are determined using $\mathbf{A}_{22}$.

Minimal Realizations

The controllable system is

$$\dot{\mathbf{x}}_c = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \mathbf{x}_c + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \tag{22}$$

$$\mathbf{y} = \begin{bmatrix} 1 & -2 \end{bmatrix} \mathbf{x}_c. \tag{23}$$

Now, we must also get rid of those modes which are unobservable

Minimal Realizations

Recipe

- Form the observability matrix, i.e.,

$$\mathbf{Q}_{\rm o} = \begin{bmatrix} \mathbf{C}^{\rm T} \\ \mathbf{C}^{\rm T}\mathbf{A} \\ \mathbf{C}^{\rm T}\mathbf{A}^2 \end{bmatrix} \tag{24}$$

- If $|\mathbf{Q}_{\rm o}| = 0$, there exist some unobservable modes.
- Take as many rows as are linearly independent, and extend from below with vectors that gives a full rank.

$$\widetilde{\mathbf{Q}}_{\rm o} = \begin{bmatrix} \mathbf{Q}_{\rm oli} \\ \text{extention} \end{bmatrix}, \tag{25}$$

  $\mathbf{Q}_{\rm oli}$ are the linear independent rows of the observability matrix.
- Use $\widetilde{\mathbf{Q}}_{\rm o}$ as $\mathbf{T}$ for a similarity transformation.

$$\mathbf{Q}_\text{o} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad |\mathbf{Q}_\text{o}| = 0, \tag{26}$$

$$\widetilde{\mathbf{Q}}_\text{o} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad |\widetilde{\mathbf{Q}}_\text{o}| = 1. \tag{27}$$

The new state-space representation is given by

$$\dot{\nu} = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \nu + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{28}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \nu. \tag{29}$$

The second state-variable does not directly go into the output neither does it influence any of the other states. Therefore, ν_2 is unobservable.

Unobservable States

In general, this transformation results in

$$\dot{\nu} = \begin{bmatrix} \mathbf{A}_{11} & 0 \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \nu + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} u. \quad (30)$$

$$y = \begin{bmatrix} \mathbf{C}_1^{\mathrm{T}} & 0 \end{bmatrix} \nu + \mathbf{D}u. \quad (31)$$

All unobservable eigenvalues are determined using $\mathbf{A}_{22}$.

There is a stable unobservable eigenvalue at $\lambda = -1$. The remaining subsystem is given by

$$\dot{\nu}_1 = -2\nu_1 + u \quad \text{and} \quad y = \nu_1. \quad (32)$$

Or

$$G(s) = \frac{1}{s+2}. \quad (33)$$

Note

While we have shown earlier that similar state-space representations always show the same transfer functions, it is incorrect to conclude that all possible representations of a system with the same transfer function are similar to each other, e.g., The third order system is not related by a similarity transformation to the first order.

$$\dot{\mathbf{x}} = \begin{bmatrix} -56 & -15 & 30 \\ 30 & 9 & -16 \\ -90 & -25 & 48 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} u \text{ and } \mathbf{y} = \begin{bmatrix} -7 & -1 & 4 \end{bmatrix} \mathbf{x}. \tag{34}$$

$$\dot{\xi} = -2\xi + u \quad \text{and} \quad y = -\xi. \tag{35}$$

Note

The presence of uncontrollable or unobservable modes leads always to pole/zero cancellation in the transfer function. It is correct (for SISO systems only) that pole/zero cancelation in the transfer function always indicate the presence of either uncontrollable or unobservable state variables.

$$G(s) = \frac{(s-4)(s+1)}{(s-4)(s+1)(s+2)}. \tag{36}$$

Note

Among the state-space representations there exists one important subsystem usually containing all these representations in which all uncontrollable and unobservable states have been eliminated. These are the representations of minimal degree and they are often referred to as minimal realizations.

$$\dot{\mathbf{x}} = \begin{bmatrix} -56 & -15 & 30 \\ 30 & 9 & -16 \\ -90 & -25 & 48 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} u, \tag{37}$$

$$\mathbf{y} = \begin{bmatrix} -7 & -1 & 4 \end{bmatrix} \mathbf{x}. \tag{38}$$

$$\dot{\nu}_1 = -2\nu_1 + u \quad \text{and} \quad y = \nu_1. \tag{39}$$

Note

If these exist unstable uncontrollable/unobservable modes, the smallest disturbance will drive some internal state-variables into saturation. Saturation is a non-linear element. The system will at this moment behave like a linear system, and the uncontrollable and observable modes become suddenly visible and will make the performance of the system unstable. There is nothing that we can do to make such systems work correctly.