Oscillation: A system oscillates when it has a nontrivial periodic solution

$$x(t+T) = x(t), \ \ \forall\, t \geq 0$$

Linear (Harmonic) Oscillator:

$$\dot{z} = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} z$$

$$z_1(t) = r_0 \cos(\beta t + \theta_0), \qquad z_2(t) = r_0 \sin(\beta t + \theta_0)$$

$$r_0 = \sqrt{z_1^2(0) + z_2^2(0)}, \qquad \theta_0 = \tan^{-1}\left[\frac{z_2(0)}{z_1(0)}\right]$$

The linear oscillation is not practical because

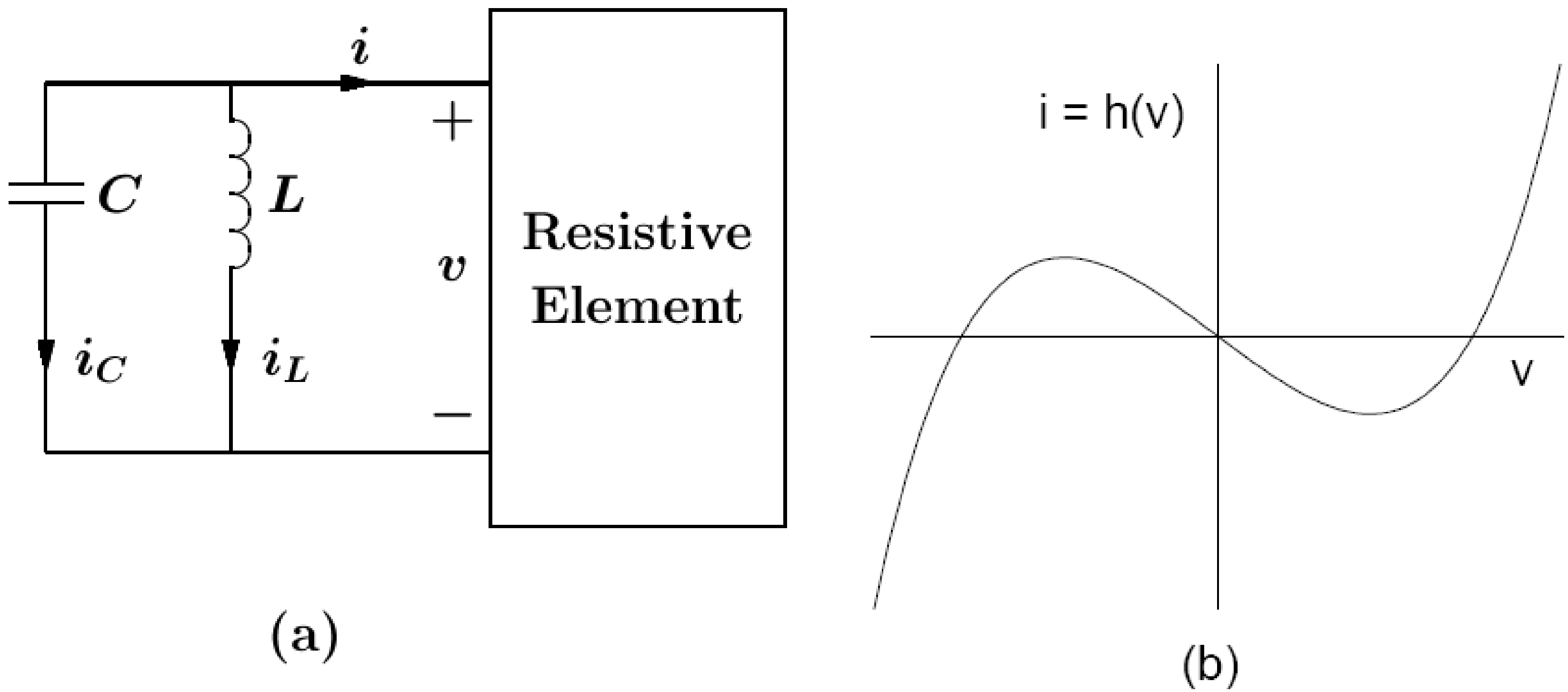
- It is not structurally stable. Infinitesimally small perturbations may change the type of the equilibrium point to a stable focus (decaying oscillation) or unstable focus (growing oscillation)

- The amplitude of oscillation depends on the initial conditions

The same problems exist with oscillation of nonlinear systems due to a center equilibrium point (e.g., pendulum without friction)

Limit Cycles:

Example: Negative Resistance Oscillator



$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \varepsilon h'(x_1) x_2 \end{aligned}$$

There is a unique equilibrium point at the origin

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon h'(0) \end{bmatrix}$$

$$\lambda^2 + \varepsilon h'(0)\lambda + 1 = 0$$

$h'(0) < 0 \;\Rightarrow$ Unstable Focus or Unstable Node

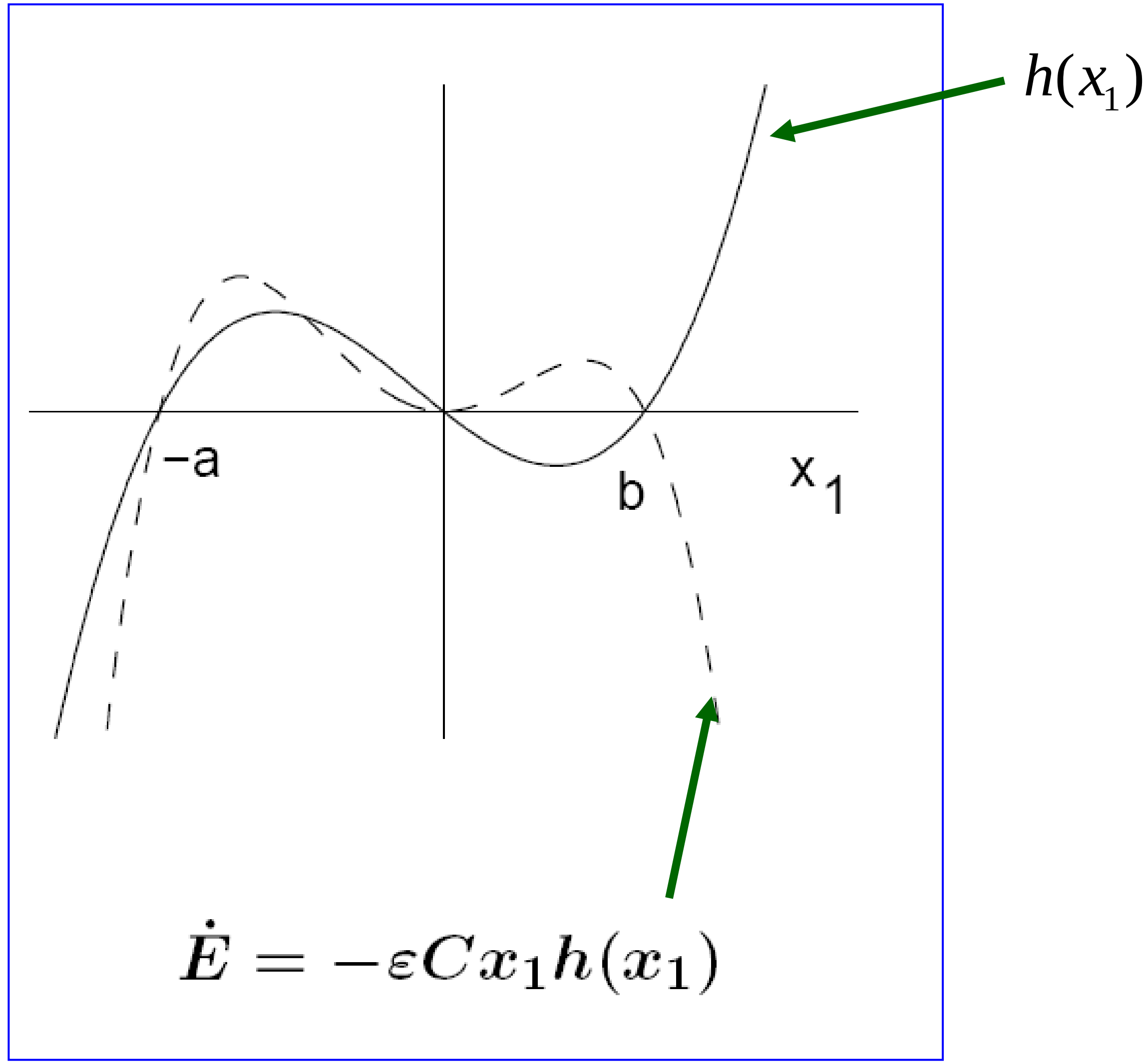
Energy Analysis:

$$E = \tfrac{1}{2} C v_C^2 + \tfrac{1}{2} L i_L^2$$

$$v_C = x_1 \quad \text{and} \quad i_L = -h(x_1) - \frac{1}{\varepsilon} x_2$$

$$E = \tfrac{1}{2} C \{x_1^2 + [\varepsilon h(x_1) + x_2]^2\}$$

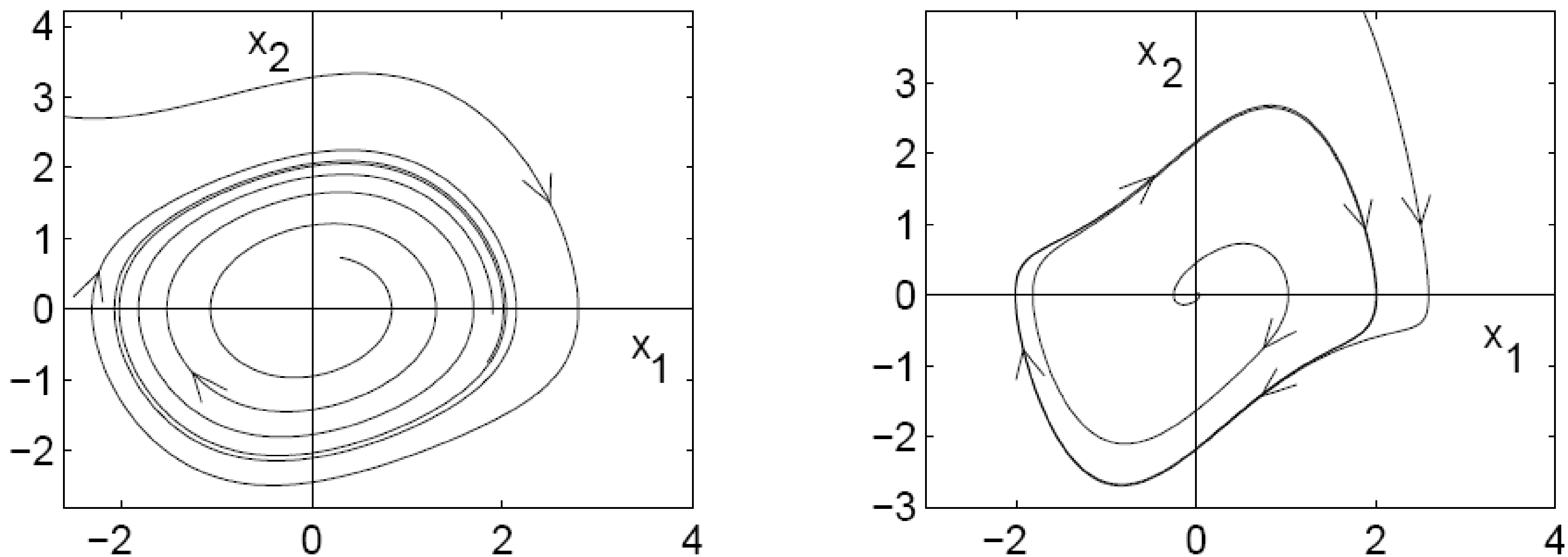
$$\begin{aligned}
\dot{E} &= C\{x_1 \dot{x}_1 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)\dot{x}_1 + \dot{x}_2]\} \\
&= C\{x_1 x_2 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1) x_2 - x_1 - \varepsilon h'(x_1) x_2]\} \\
&= C[x_1 x_2 - \varepsilon x_1 h(x_1) - x_1 x_2] \\
&= -\varepsilon C x_1 h(x_1)
\end{aligned}$$



$h(x_1)$
$-a$
b
x_1
$\dot{E} = -\varepsilon C x_1 h(x_1)$

Example: Van der Pol Oscillator

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2
\end{aligned}
$$

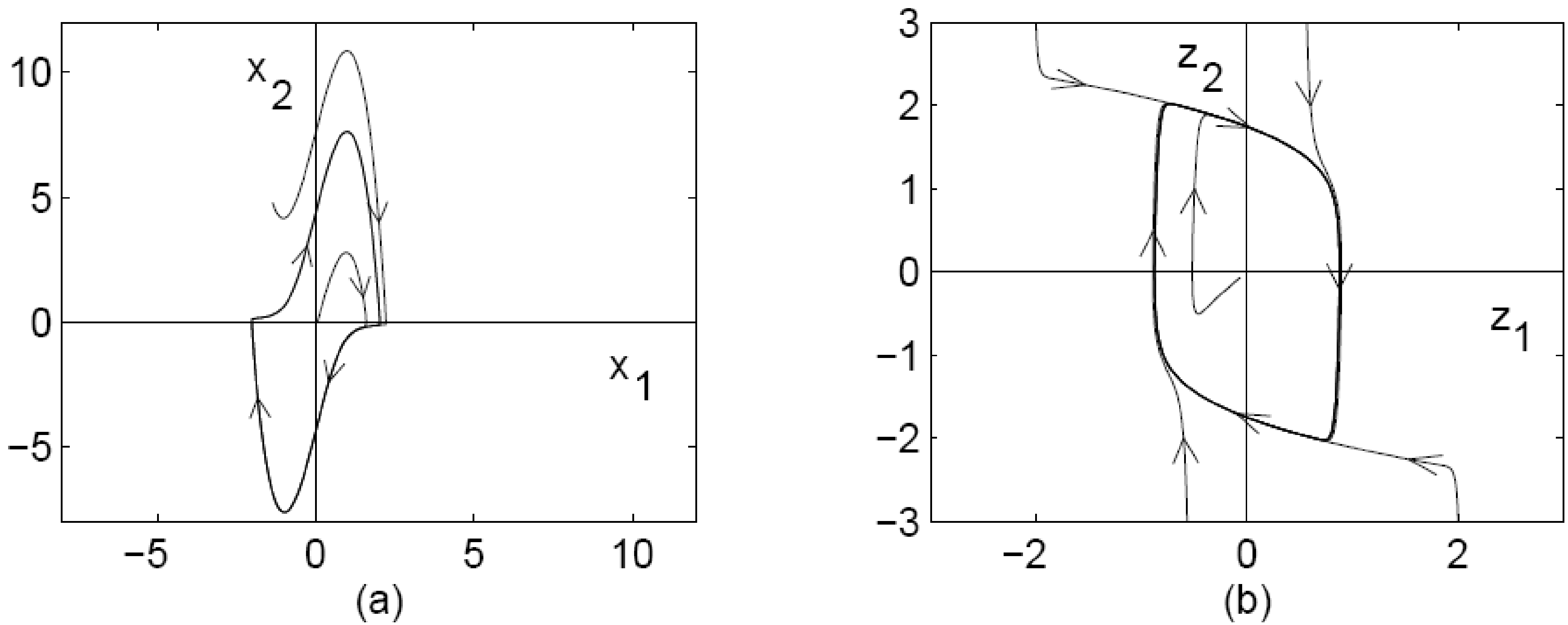


(a) (b)

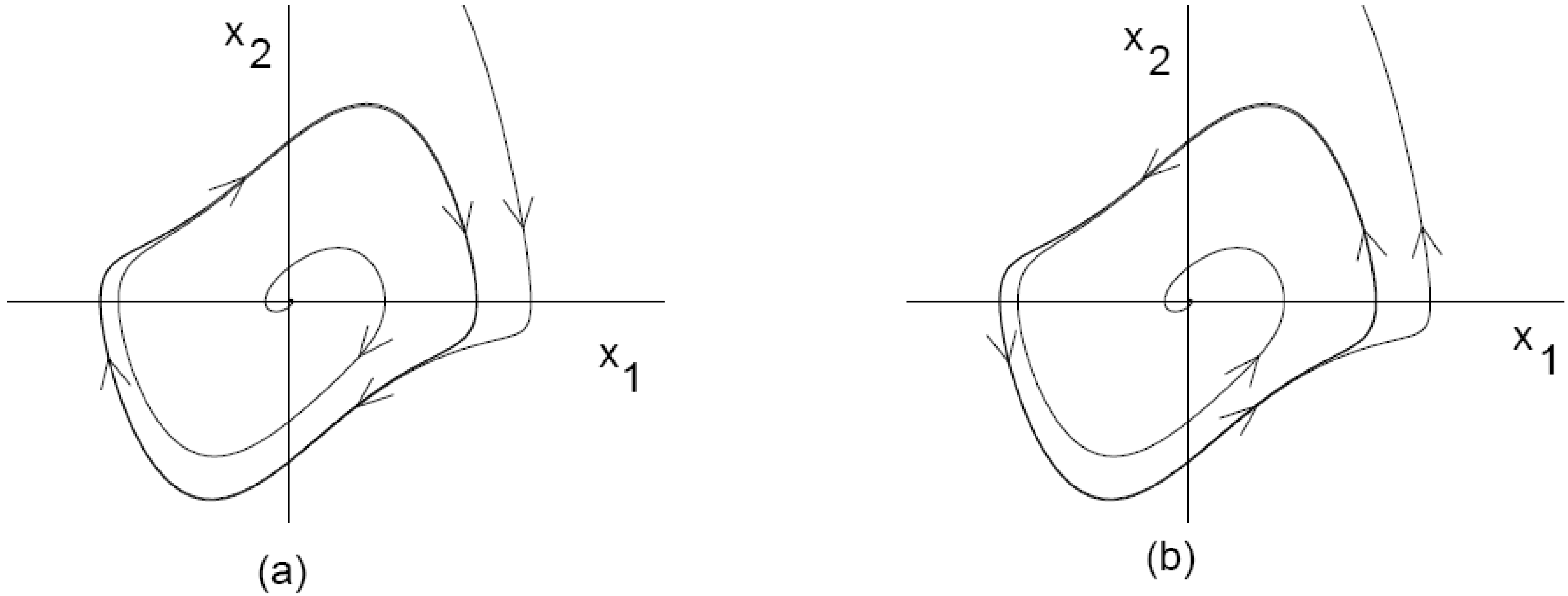
$\varepsilon = 0.2$ $\varepsilon = 1$

$$\dot{z}_1 = \frac{1}{\varepsilon} z_2$$

$$\dot{z}_2 = -\varepsilon(z_1 - z_2 + \tfrac{1}{3} z_2^3)$$



$\varepsilon = 5$



Stable Limit Cycle

Unstable Limit Cycle

Existence of Periodic Orbits

$$\dot{x} = f(x) \qquad (2.7)$$

Lemma 2.1 (Poincaré–Bendixson Criterion) *Consider the system (2.7) and let M be a closed bounded subset of the plane such that*

- *M contains no equilibrium points, or contains only one equilibrium point such that the Jacobian matrix $[\partial f/\partial x]$ at this point has eigenvalues with positive real parts. (Hence, the equilibrium point is unstable focus or unstable node.)*
- *Every trajectory starting in M stays in M for all future time.*

Then, M contains a periodic orbit of (2.7).

Lemma 2.2 (Bendixson Criterion) *If, on a simply connected region[17] D of the plane, the expression $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ is not identically zero and does not change sign, then system (2.7) has no periodic orbits lying entirely in D.* ◇

Bifurcation

Bifurcation is a change in the equilibrium points or periodic orbits, or in their stability properties, as a parameter is varied

Example

$$\begin{aligned} \dot{x}_1 &= \mu - x_1^2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

Find the equilibrium points and their types for different values of μ

For $\mu > 0$ there are two equilibrium points at $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$

Linearization at $(\sqrt{\mu}, 0)$:

$$\begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

$(\sqrt{\mu}, 0)$ is a stable node

Linearization at $(-\sqrt{\mu}, 0)$:

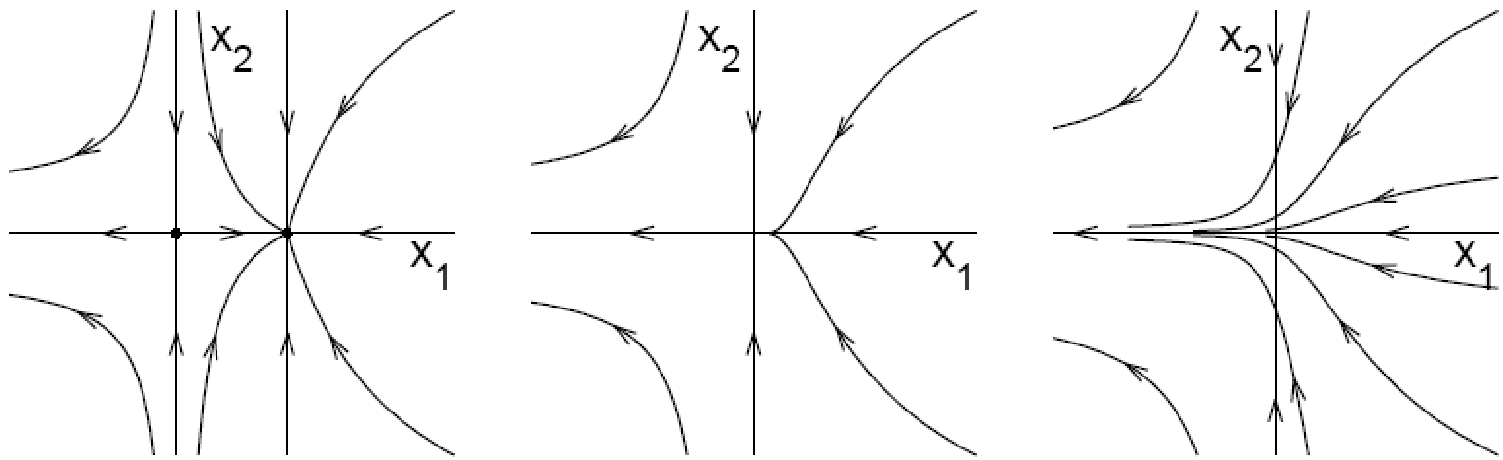
$$\begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

$(-\sqrt{\mu}, 0)$ is a saddle

$$\dot{x}_1 = \mu - x_1^2, \qquad \dot{x}_2 = -x_2$$

No equilibrium points when $\mu < 0$

As μ decreases, the saddle and node approach each other, collide at $\mu = 0$, and disappear for $\mu < 0$



$\mu > 0$ $\mu = 0$ $\mu < 0$

μ is called the bifurcation parameter and $\mu = 0$ is the bifurcation point

Bifurcation Diagram

(a) Saddle−node bifurcation

Example

$$\dot{x}_1 = \mu x_1 - x_1^2, \qquad \dot{x}_2 = -x_2$$

Two equilibrium points at $(0,0)$ and $(\mu,0)$

The Jacobian at $(0,0)$ is $\begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}$

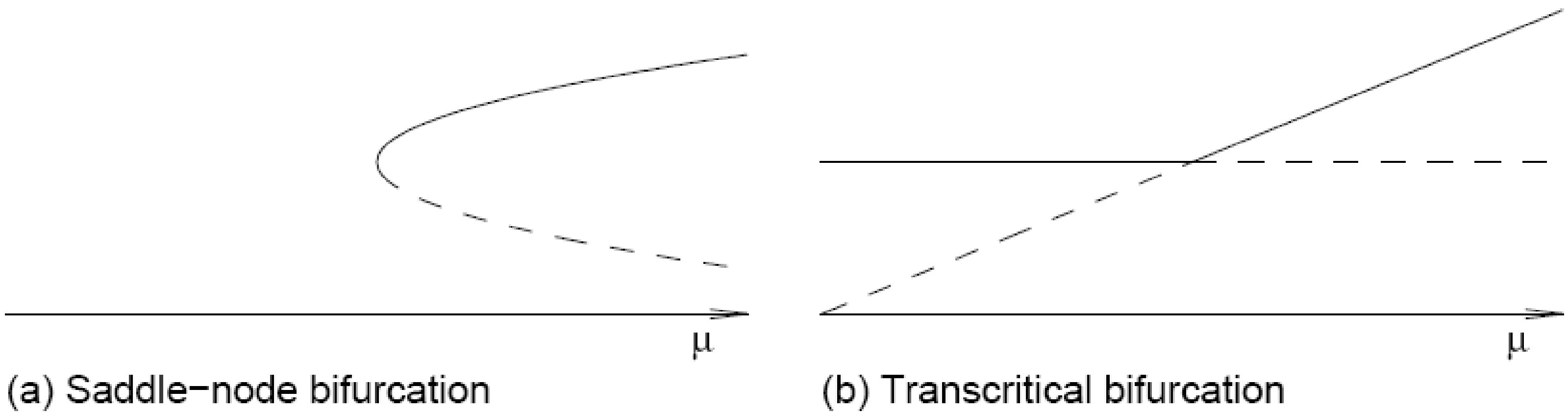
$(0,0)$ is a stable node for $\mu < 0$ and a saddle for $\mu > 0$

The Jacobian at $(\mu,0)$ is $\begin{bmatrix} -\mu & 0 \\ 0 & -1 \end{bmatrix}$

$(\mu,0)$ is a saddle for $\mu < 0$ and a stable node for $\mu > 0$

An eigenvalue crosses the origin as μ crosses zero

While the equilibrium points persist through the bifurcation point $\mu = 0$, $(0, 0)$ changes from a stable node to a saddle and $(\mu, 0)$ changes from a saddle to a stable node



dangerous or hard

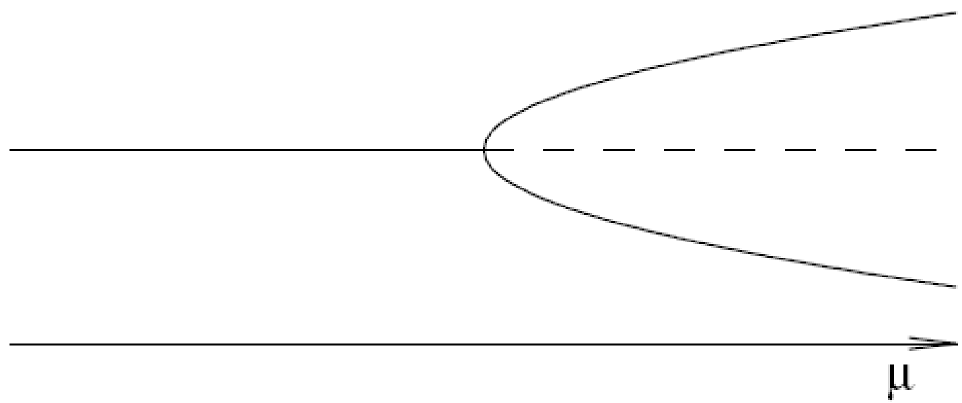
safe or soft

Example

$$\dot{x}_1 = \mu x_1 - x_1^3, \qquad \dot{x}_2 = -x_2$$

For $\mu < 0$, there is a stable node at the origin

For $\mu > 0$, there are three equilibrium points: a saddle at $(0, 0)$ and stable nodes at $(\sqrt{\mu}, 0)$, and $(-\sqrt{\mu}, 0)$



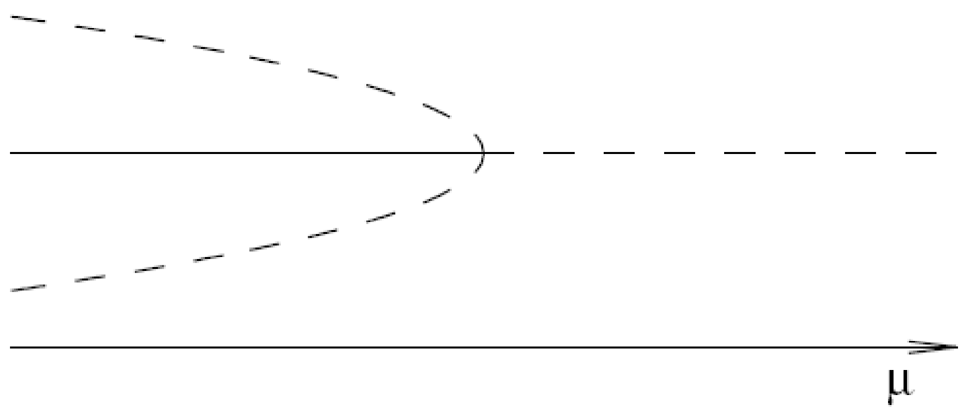
(c) Supercritical pitchfork bifurcation

Example

$$\dot{x}_1 = \mu x_1 + x_1^3, \qquad \dot{x}_2 = -x_2$$

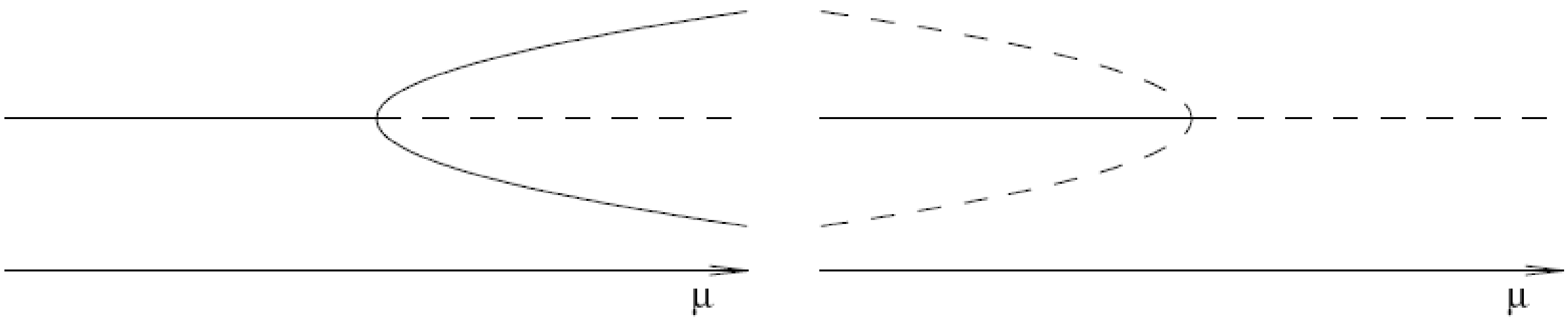
For $\mu < 0$, there are three equilibrium points: a stable node at $(0, 0)$ and two saddles at $(\pm\sqrt{-\mu}, 0)$

For $\mu > 0$, there is a saddle at $(0, 0)$



(d) Subcritical pitchfork bifurcation

Notice the difference between supercritical and subcritical pitchfork bifurcations



(c) Supercritical pitchfork bifurcation

(d) Subcritical pitchfork bifurcation

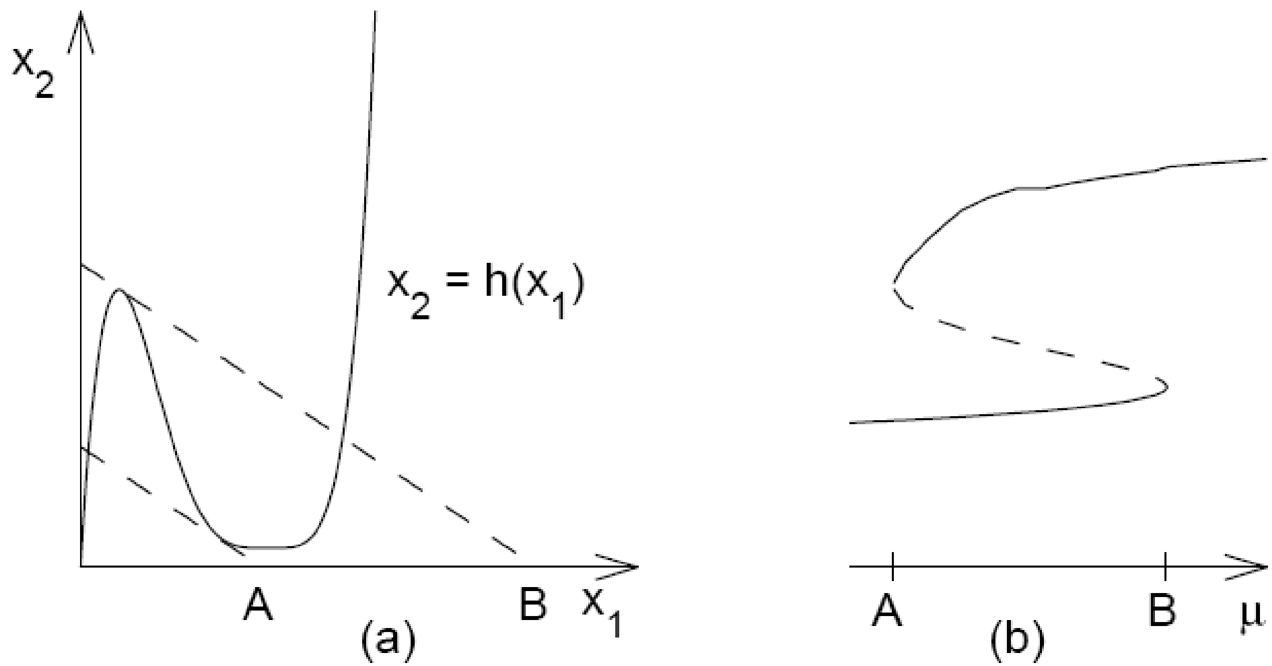
safe or soft

dangerous or hard

Example: Tunnel diode Circuit

$$\dot{x}_1 = \frac{1}{C}\left[-h(x_1) + x_2\right]$$

$$\dot{x}_2 = \frac{1}{L}\left[-x_1 - Rx_2 + \mu\right]$$



Example

$$\begin{aligned}\dot{x}_1 &= x_1(\mu - x_1^2 - x_2^2) - x_2 \\ \dot{x}_2 &= x_2(\mu - x_1^2 - x_2^2) + x_1\end{aligned}$$

There is a unique equilibrium point at the origin

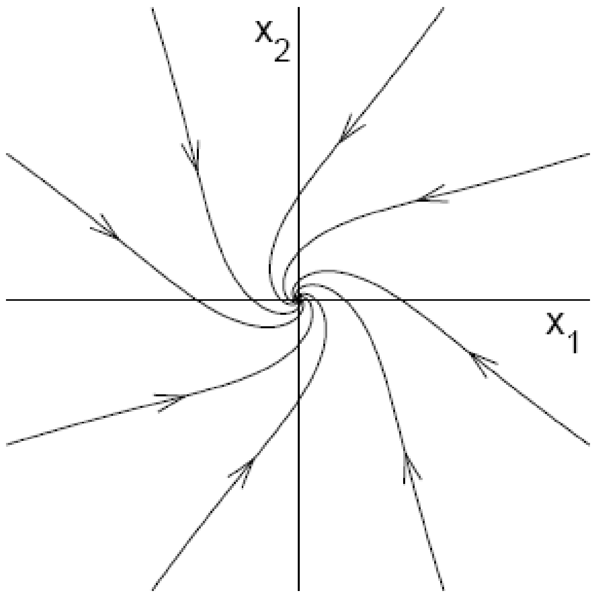
$$\text{Linearization:} \quad \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

Stable focus for $\mu < 0$, and unstable focus for $\mu > 0$

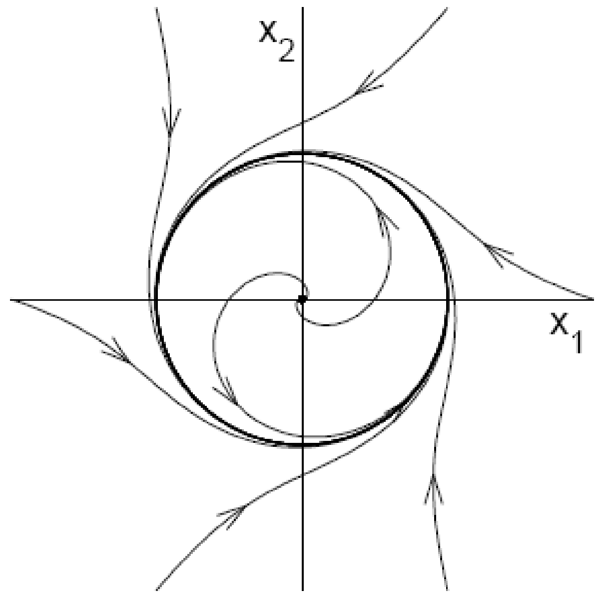
A pair of complex eigenvalues cross the imaginary axis as μ crosses zero

$$\dot{r} = \mu r - r^3 \quad \text{and} \quad \dot{\theta} = 1$$

For $\mu > 0$, there is a stable limit cycle at $r = \sqrt{\mu}$



$\mu < 0$



$\mu > 0$

Supercritical Hopf bifurcation

Example

$$\begin{aligned}\dot{x}_1 &= x_1\left[\mu + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2\right] - x_2 \\ \dot{x}_2 &= x_2\left[\mu + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2\right] + x_1\end{aligned}$$

There is a unique equilibrium point at the origin

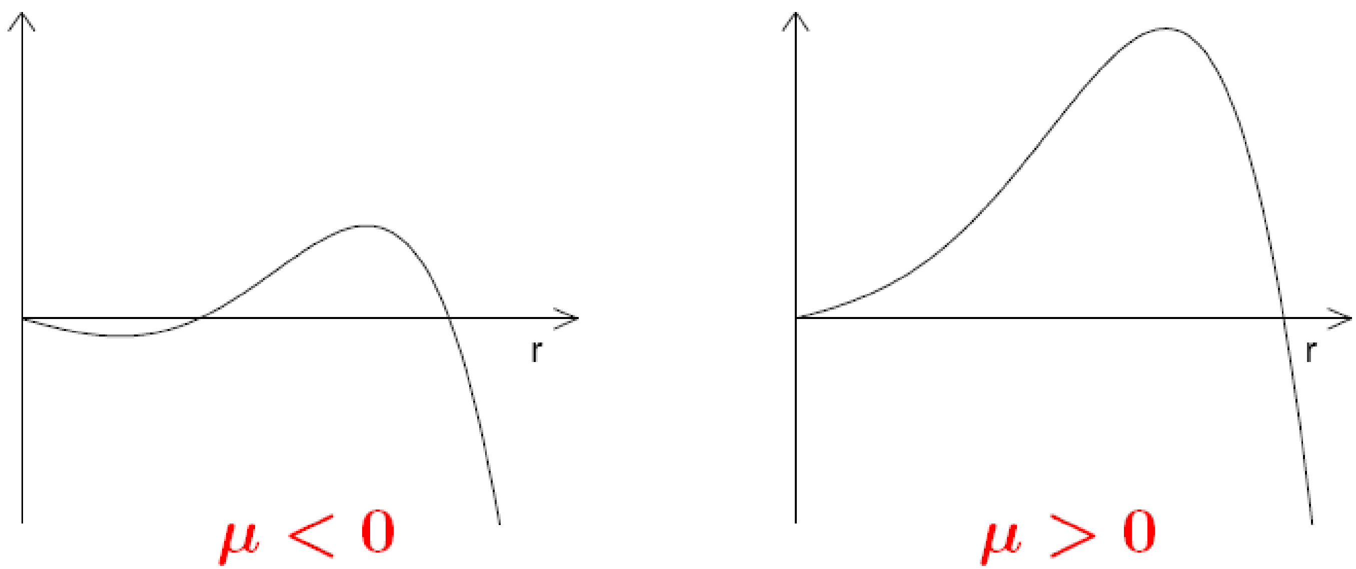
Linearization: $\begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$

Stable focus for $\mu < 0$, and unstable focus for $\mu > 0$

A pair of complex eigenvalues cross the imaginary axis as μ crosses zero

$$\dot{r} = \mu r + r^3 - r^5 \quad \text{and} \quad \dot{\theta} = 1$$

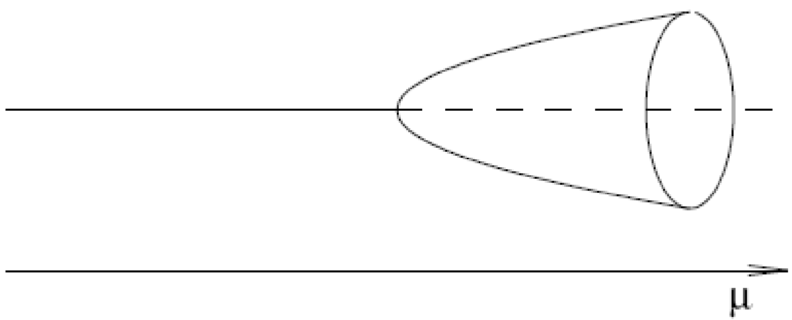
Sketch of $\mu r + r^3 - r^5$:



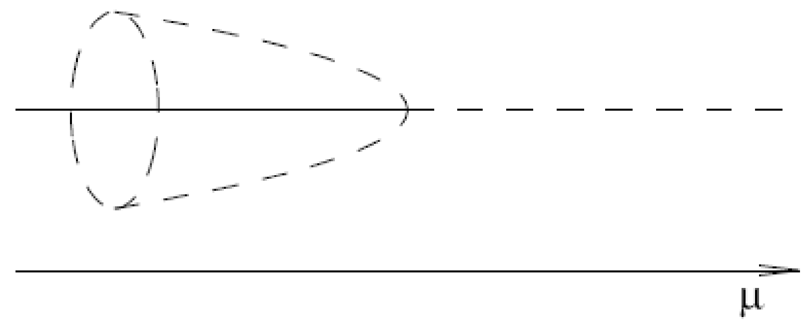
For small $|\mu|$, the stable limit cycles are approximated by $r = 1/\sqrt{2}$, while the unstable limit cycle for $\mu < 0$ is approximated by $r = \sqrt{|\mu|}$

As μ increases from negative to positive values, the stable focus at the origin merges with the unstable limit cycle and bifurcates into unstable focus

Subcritical Hopf bifurcation



(e) Supercritical Hopf bifurcation



(f) Subcrtitical Hopf bifurcation

safe or soft

dangerous or hard

All six types of bifurcation occur in the vicinity of an equilibrium point. They are called local bifurcations

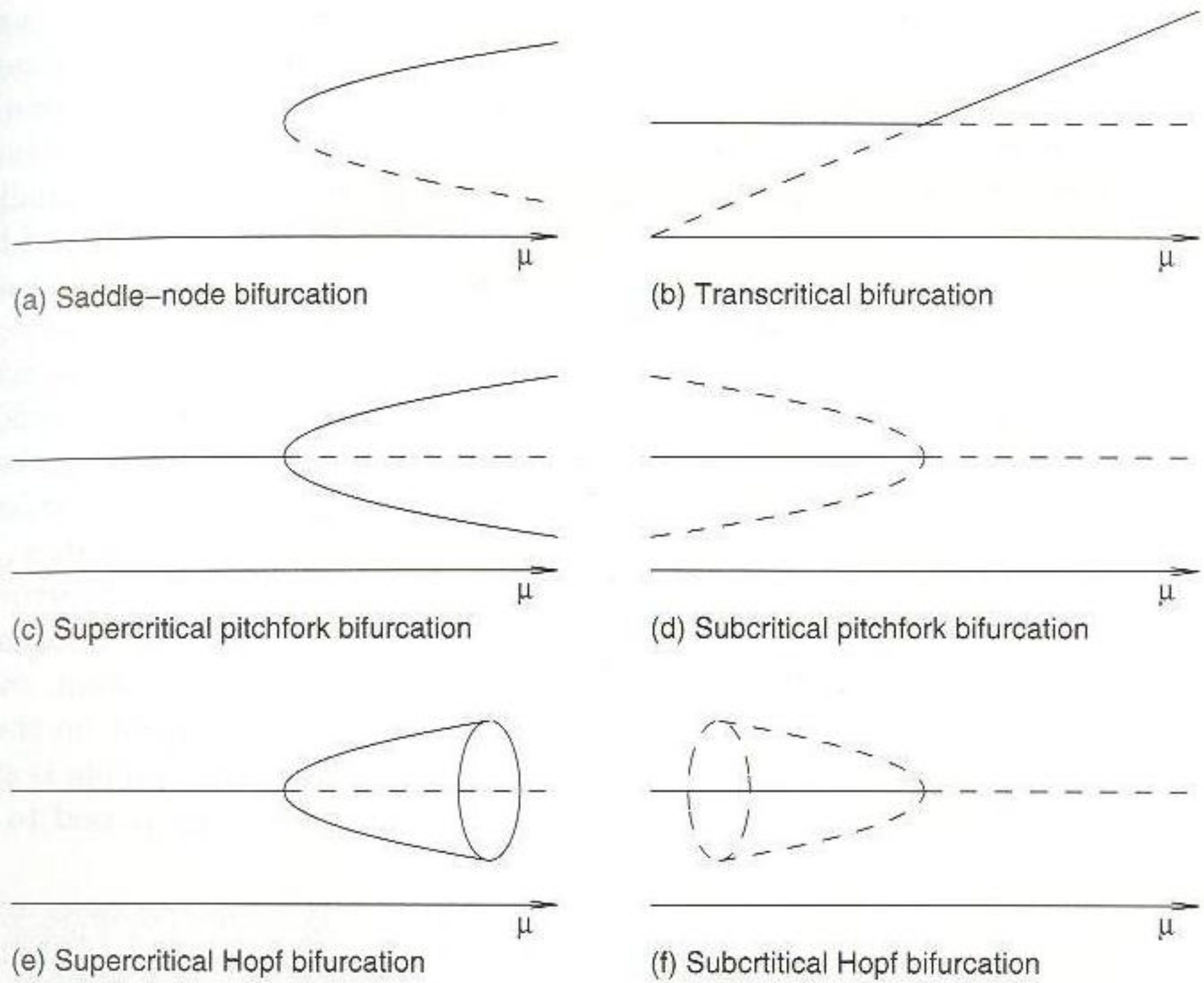


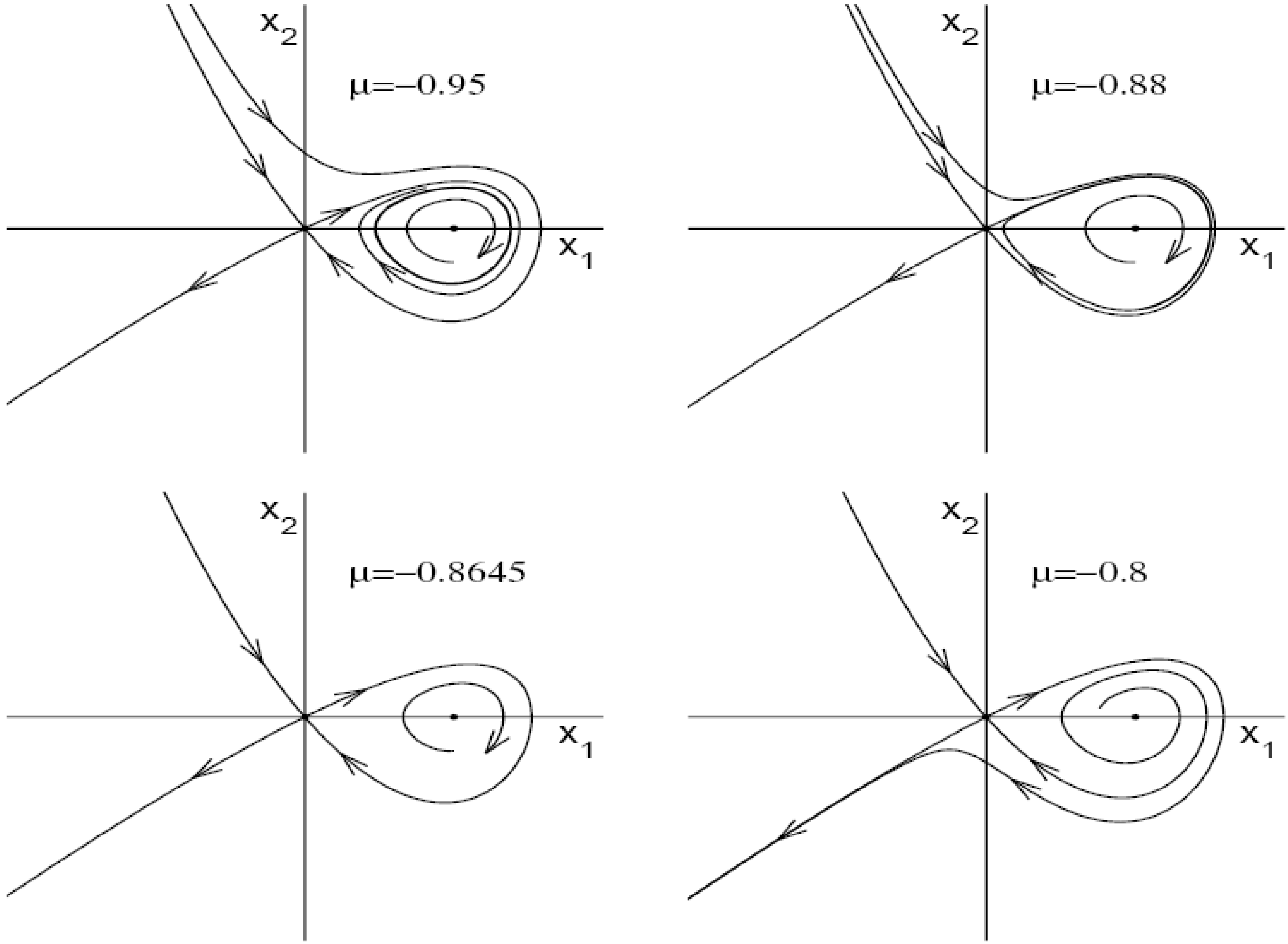
Figure 2.28: Bifurcation diagrams.

Example of Global Bifurcation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mu x_2 + x_1 - x_1^2 + x_1 x_2 \end{aligned}$$

There are two equilibrium points at $(0,0)$ and $(1,0)$. By linearization, we can see that $(0,0)$ is always a saddle, while $(1,0)$ is an unstable focus for $-1 < \mu < 1$

Limit analysis to the range $-1 < \mu < 1$



Saddle–connection (or homoclinic) bifurcation