Types of Sampled Data Systems

- Sampled signals are easier to transmit.
- Transmitted signals can be regenerated without transmission error.
- Sampled signals can more easily be coded (cryptology).
- Sampled signals can better be modulated (multiplexing).

Figure: Sampling.
Types of Sampled Data Systems

Figure: Sampling.

![Diagram of sampled data system with feedback loop and components labeled: sample, ZOH, continuous controller, continuous plant, and output y(t).]
A ZOH samples the current signal and holds that value until the next sample.

In most systems, it is difficult to generate and transmit narrow, large amplitude pulses.

We can often use a variety of filtering and interpolation techniques to reconstruct the original time-domain signal, however often the zero-order hold signal is sufficiently accurate.
Given \( f(t) \) ⇒ what is \( f^*(s) \)?

**Example**

\[
F(t) = \varepsilon(t) \quad \Rightarrow \quad F(s) = \frac{1}{s}
\]

\[
f^*(t) = \sum_{k=0}^{\infty} f(kt)\delta(t - kT) = \sum_{k=0}^{\infty} \delta(t - kT)
\]

\[
f^*(s) = \sum_{k=0}^{\infty} \exp^{-kTs} = 1 + \exp^{-Ts} + \exp^{-2Ts} + \ldots
\]

\[
= \frac{1}{1 - \exp^{-Ts}}
\]

Is there a simple way to compute \( f^*(s) \) out of \( f(s) \)?
Is there a simple way to compute $F^*(s)$ out of $F(s)$?

For poles of $F(s)$ with multiplicity 1 (simple poles):

\[ F(s) = \frac{P(s)}{Q(s)} \Rightarrow F^*(s) = \sum_{n=1}^{k} \frac{P(s_n)}{Q'(s_n)} \frac{1}{1 - \exp^{-T(s-s_n)}} \]

$k$ is the number of poles.

**Example:** $F(s) = \frac{1}{s}$ \Rightarrow One pole at zero

\[ \Rightarrow P(s) = 1 , \; Q(s) = s \Rightarrow Q'(s) = 1 \]

\[ F^*(s) = \sum_{n=1}^{1} \frac{1}{1} \frac{1}{1 - \exp^{-T(s-0)}} = \frac{1}{1 - \exp^{-Ts}} \]
Sampling of Arbitrary Signals

\[ F(s) = \frac{1}{s(s+1)} \Rightarrow \text{two poles, one at zero and one at -1} \]

\[ F^*(s) = \frac{P(s_1)}{Q'(s_1)} \frac{1}{1 - \exp^{-T(s-s_1)}} + \frac{P(s_2)}{Q'(s_2)} \frac{1}{1 - \exp^{-T(s-s_2)}} \]

\[ = \frac{1}{1 - \exp^{-Ts}} \frac{1}{1 - \exp^{-T(s+1)}} \]

\[ = \frac{1 - \exp^{-T} \exp^{-Ts} - 1 + \exp^{-Ts}}{(1 - \exp^{-Ts})(1 - \exp^{-T} \exp^{-Ts})} \]

\[ = \frac{(1 - \exp^{-T}) \exp^{-Ts}}{(1 - \exp^{-Ts})(1 - \exp^{-T} \exp^{-Ts})} \]

It turns out that, in both examples, \( F^*(s) \) is a function of the term \( \exp^{-Ts} \).
The rule is pretty obvious and easy to derive. We use the partial fraction expansion on $F(s)$:

$$F(s) = \sum_{n=1}^{k} \frac{P(s_n)}{Q'(s_n)} \frac{1}{s - s_n}$$

when going from $F(s) \Rightarrow F^*(s)$, we need only to find $F^*(s)$ for \( \frac{a}{s-b} \), and then we know the general formula for all $F(s)$ with simple poles

$$F(s) = \frac{a}{s-b} \quad \Rightarrow \quad F^*(s) = \frac{a}{1 - \exp^{-T(s-b)}}$$

as can be easily verified by looking at the infinite series.
Similar for multiple poles (multiplicity of poles $n$ is $m_n$):

$$F^*(s) = \sum_{n=1}^{k} \sum_{i=1}^{m_n} \frac{(-1)^{m_n-1} K_{ni}}{(m_n - 1)!} \frac{\partial^{m_n-i} \Delta T(s)}{\partial s^{m_n-i}} \bigg|_{s=s-s_n}$$

where

$$K_{ni} = \frac{1}{(i - 1)!} \frac{\partial^{i-1} [(s - s_n)^{m_n} F(s)]}{\partial s^{i-1}} \bigg|_{s=s_n}$$

and

$$\Delta T(s) = \frac{1}{1 - \exp^{-Ts}}$$
Example

\[ F(s) = \frac{2}{(s + a)^3} \Rightarrow s_1 = -a \quad ; \quad m_1 = 3 \]

\[ K_{11} = \frac{1}{1} (s + a)^3 F(s) \mid_{s=-a} = 2 \mid_{s=-a} = 2 \]

\[ K_{12} = \frac{1}{1} \frac{\partial}{\partial s} (2) \mid_{s=-a} = 0 \]

\[ K_{13} = \frac{1}{2} \frac{\partial^2(2)}{\partial s^2} \mid_{s=-a} = 0 \]

only the first term gives a contribution

\[ F^*(s) = \frac{(-1)^2 2}{(3 - 1)!} \frac{\partial \Delta T(s)}{\partial s^2} \mid_{s=s-s_n} \]
Sampling of Arbitrary Signals

\[ F^*(s) = \frac{(-1)^2}{(3 - 1)!} \frac{\partial \Delta T(s)}{\partial s^2} \bigg|_{s = s - s_n} \]

\[ \frac{\partial}{\partial s} \Delta T(s) = \frac{\partial}{\partial s} \left( \frac{1}{1 - \exp^{-Ts}} \right) = \frac{-T \exp^{-Ts}}{(1 - \exp^{-Ts})^2} \]

\[ \frac{\partial^2}{\partial s^2} \Delta T(s) = \frac{\partial}{\partial s} \left( \frac{\partial}{\partial s} \Delta T(s) \right) = \frac{\partial}{\partial s} \left( \frac{-T \exp^{-Ts}}{(1 - \exp^{-Ts})^2} \right) \]

\[ \frac{\partial^2}{\partial s^2} \Delta T(s) = \frac{T^2 \exp^{-Ts}(1 + 2 \exp^{-Ts})}{(1 - \exp^{-Ts})^3} \]

evaluate at \( s = s - s_n = s + a \)

\[ F^*(s) = \frac{T^2 e^{-T(s+a)}(1 + 2e^{-T(s+a)})}{(1 - e^{-T(s+a)})^3} = \frac{T^2 e^{-Ts} e^{-aT}(1 + 2e^{-aT} e^{-Ts})}{(1 - e^{-aT} e^{-Ts})^3} \]
Sampling of Arbitrary Signals

\[ F^*(s) = \frac{T^2 e^{-Ts} (1 + 2e^{-T(s+a)})}{(1 - e^{-T(s+a)})^3} = \frac{T^2 e^{-Ts} e^{-aT} (1 + 2e^{-aT} e^{-Ts})}{(1 - e^{-aT} e^{-Ts})^3} \]

Again we obtained an expression in \( e^{-Ts} \) \textbf{Let us set:}
\[
z = e^{Ts} \iff z^{-1} = e^{-Ts}
\]

\[ F^*(s) = \tilde{F}(z^{-1}) = \frac{T^2 z^{-1} e^{-aT} (1 + 2e^{-aT} z^{-1})}{(1 - e^{-aT} z^{-1})^3} = F(z) = \frac{T^2 ze^{-aT} (z + 2e^{-aT})}{(z - e^{-aT})^3} \]
Sampling of Arbitrary Signals

Laplace Transform

\[ f(t) \implies F(s) \quad \text{Laplace} \]

\[ f(t) \implies \tilde{F}(s^{-1}) \quad \text{Inverse Laplace} \]

Z-Transform

\[ f(t) \implies F(z) \quad \text{Z-Transform} \]

\[ f(t) \implies \tilde{F}(z^{-1}) \quad \text{Inverse Z-Transform} \]
Like the Laplace transform, also the z-transform has a physical meaning:

<table>
<thead>
<tr>
<th>Physical Meaning</th>
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<tbody>
<tr>
<td>$s = \text{Derivative } \left( \frac{d}{dt} \right)$</td>
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Sampling of Arbitrary Signals

Physical Meaning

\[ s = \text{Derivative} \left( \frac{d}{dt} \right) \]

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\[ z = \text{Left shifting by} \ T \]

\[ \frac{1}{z} = z^{-1} = \text{Right shifting by} \ T \]
The signal got delayed by $T$. Like in the Laplace transform, physically requires higher order denominator than nominators.
Difference Equation

\[ g(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \ldots + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_0} = \sum_{k=0}^{\infty} g_k z^{-k} \]

Comparison of the coefficients leads to the recursive formula:

\[ g_k = b_{n-k} - a_{n-1} g_{k-1} - a_{n-2} g_{k-2} - \ldots - a_0 g_{k-n} \]

where \( b_k, g_k = 0 \).

**Example**

\[ g(z) = \frac{z^3 + 2z^2 + 3z}{z^4 - 1} \]

\[ g_0 = b_4 - a_3 g_{-1} - \ldots = b_4 = 0 \]
\[ g_1 = b_3 - a_3 g_0 = 1 - 0 = 1 \]
\[ g_2 = b_2 - a_3 g_1 - a_2 g_0 = 2 - 0 - 0 = 2 \]
\[ g_3 = b_1 - a_3 g_2 - a_2 g_1 - a_1 g_0 = 3 - 0 - 0 - 0 = 3 \]
\[ g_4 = b_0 - a_3 g_3 - a_2 g_2 - a_1 g_1 - a_0 g_0 \]
\[ g(z) = \frac{z^3 + 2z^2 + 3z}{z^4 - 1} = \frac{y(z)}{u(z)} \]

\[ z^4y(z) - y(z) = z^3u(z) + 2z^2u(z) + 3zu(z) \]

Now apply the shifting property

\[ y^*(t + 4T) - y^*(t) = u^*(t + 3T) + 2u^*(t + 2T) + 3u^*(t + T) \]

or

\[ y^*(t + T) = y^*(t - 3T) + u^*(t) + 2u^*(t - T) + 3u^*(t - 2T) \]

Given a system with the z-transfer function \( g(z) \) we can simulate the behaviour of the output out of measurements of previous inputs and outputs.
Difference Equation

Given

\[ g(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \ldots + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_0} \]

We can obtain

\[ x(k + 1) = Fx(k) + gu(k) \]
\[ y(k) = hx(k) + iu(k) \]

**F**: System matrix  
**g**: Input vector  
**h**: Output vector  
**i**: Direct coupling.
Relation Between $\omega$-Domain and Frequency Domain

$\omega$-Domain

$$H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k}$$

$z$-Domain

$$H(z) = \sum_{k=0}^{M} b_k z^k$$

Comparing the above we see that the connection $s$ setting $z = e^{j\omega}$ in $H(z)$, i.e.,

$$H(e^{j\omega}) = H(Z) \mid_{z=e^{j\omega}}$$
If we consider that z-plane, we see that $H(e^{j\omega})$ corresponds to evaluating $H(z)$ on the unit circle.
The z-Plane and the Unit Circle

From this interpretation we also can see why \( H(e^{j\omega}) \) is periodic with period \( 2\pi \). As \( \omega \) increases it continues to sweep around the unit circle over and over again.
The Zeros and Poles of $H(z)$

$$H(Z) = \frac{(Z - Z_1)(Z - Z_2)(Z - Z_3)}{Z^3}$$

- The **zeros** are the locations where $H(z) = 0$, i.e., $z_1, z_2, z_3$
- The **poles** are where $H(z) \to \infty$, i.e., $z \to 0$
- The **poles** and **zeros** only determine $H(z)$ for certain parameters
- A **pole-zero** displays the **pole** and **zero** locations in the $z$-plane.
A method for study of linear constant discrete systems is:

- Compute the transfer function of the system $H(z)$.
- Compute the transform of the input signal, $E(z)$.
- From the product, $E(z)H(z)$, which is the transform of the output signal, $U$.
- Invert the transform to obtain $u(kT)$.
The unit pulse is defined by

\[ e_1(k) = \begin{cases} 1 & (k = 0) \\ 0 & (k \neq 0) \\ \delta_k \end{cases} \]

Therefore we have

\[ E_1(z) = \sum_{-\infty}^{\infty} \delta_k z^{-k} = z^0 = 1 \]

This result is much like the continuous case, wherein the Laplace transform of the unit impulse is the constant 1.0. The quantity \( E_1(z) \) gives us an instantaneous method of relate signals to systems: to characterize the system \( H(z) \), consider the signal \( u(k) \), which is the unit impulse response, then \( U(z) = H(z) \).
Consider the unit step function defined by

\[ e_2(k) = \begin{cases} 
1 & (k \geq 0) \\
0 & (k < 0) \\
1(k) & 
\end{cases} \]

In this case, the z-transform is

\[ E_2(z) = \sum_{k=-\infty}^{\infty} e_2(k)z^{-k} = \sum_{k=0}^{\infty} z^{-k} \]

\[ = \frac{1}{1 - z^{-1}} \quad (| z^{-1} | < 1) \]

\[ = \frac{z}{z - 1} \quad (| z | > 1) \]

Here the transform is characterized by a zero at \( z = 0 \) and a pole at \( z = 1 \).
The Unit Step

\[ E_2(z) = \sum_{k=-\infty}^{\infty} e_2(k)z^{-k} = \sum_{k=0}^{\infty} z^{-k} \]

\[ = \frac{1}{1 - z^{-1}} \quad (|z^{-1}| < 1) \]

\[ = \frac{z}{z - 1} \quad (|z| > 1) \]

The Laplace transform of the unit step is \(1/s\); we may this keep in mind that a pole at \(s = 0\) for a continuous signal corresponds in some way to a pole at \(z = 1\) for discrete signals.
The one-sided exponential in time is

\[ e_3(k) = r^k \quad (k \geq 0) \]
\[ = 0 \quad (k < 0) \]

Now we get

\[
E_3(z) = \sum_{k=0}^{\infty} r^k z^{-k}
\]
\[
= \sum_{k=0}^{\infty} (rz^{-1})^k
\]
\[
= \frac{1}{1 - rz^{-1}} \quad (|rz^{-1}| < 1)
\]
\[
= \frac{z}{z - r} \quad (|z| > |r|)
\]
Exponential

\[ E_3(z) = \frac{1}{1 - rz^{-1}} \quad (|rz^{-1}| < 1) \]

\[ = \frac{z}{z - r} \quad (|z| > |r|) \]

The pole of \( E_3(z) \) is at \( z = r \). We also know that \( e_3(k) \) grows without bound if \( |r| > 1 \). We conclude that a \( z \)-transform that converges for large \( z \) and has a real pole outside the circle \( |z| = 1 \) corresponds to a growing signal.

**Figure:** Exponential - \( e_3(k) \) for the stable value \( r = 0.6 \).
Consider the sinusoid $e_4(k) = [r^k \cos(k\theta)]1(k)$, where we assume $r \geq 0$. We can decompose $e_4(k)$ into the sum of two complex exponentials as

$$e_4(k) = r^k \left( \frac{e^{jk\theta} + e^{-jk\theta}}{2} \right) 1(k)$$

and because the $z$-transform is linear, we need only to compute the transform of each signal complex exponential and add the results later. We thus take first

$$e_5(k) = r^k e^{jk\theta} 1(k)$$

and compute

$$E_5(z) = \sum_{k=0}^{\infty} r^k e^{jk\theta} z^{-k}$$

$$= \sum_{k=0}^{\infty} \left( re^{j\theta} z^{-1} \right)^k$$
The signal $e_5(k)$ grows without bound as $k$ gets large if and only if $r > 1$, and a system with this pulse response is BIBO stable if and only if $|r| < 1$. The boundary of stability is the unit circle.
To complete the argument given before \( e_4(k) = r^k \cos(k\theta)1(k) \), we see immediately that the other half is found by replacing \( \theta \) by \(-\theta\)

\[
\mathcal{Z}\{r^k e^{-j\theta k}1(k)\} = \frac{z}{z - re^{-j\theta}} \quad (|z| > r)
\]

and thus

\[
E_4(z) = \frac{1}{2} \left( \frac{z}{z - re^{j\theta}} + \frac{z}{z - re^{-j\theta}} \right)
\]

\[
= \frac{z(z - r \cos \theta)}{z^2 - 2r(\cos \theta)z + r^2} \quad (|z| > r)
\]

Figure: Exponential - \( e_4(k) \) for \( r = 0.7 \) and \( \theta = 45 \) deg
Figure: z-plane
Questions please