Robotics

Islam S. M. Khalil

German University in Cairo

18 October, 2014

Outline

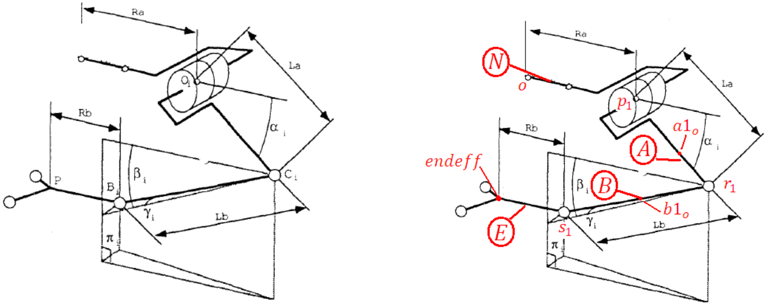
- Closed-kinematic robotics systems
- Kinematics of the Delta robot
- Dynamics of the Delta robot
- Optimization of the configuration of the Delta robot
- Optimization of the control inputs of the Delta robot

Figure: Delta robot.

Closed-Kinematic Robotics Systems

Figure: Delta robot.

Kinematics of The Delta Robot



(a) Single loop of the Delta robot (b) Frames of the Delta robot

Figure: Kinematics of a delta robot.

Kinematics of The Delta Robot

$$\mathbf{r}^{op_1} + \mathbf{r}^{p_1 r_1} + \mathbf{r}^{r_1 s_1} + \mathbf{r}^{s_1 endeff} + \mathbf{r}^{endeffo} = 0 \quad (1)$$
$$\mathbf{r}^{op_2} + \mathbf{r}^{p_2 r_2} + \mathbf{r}^{r_2 s_2} + \mathbf{r}^{s_2 endeff} + \mathbf{r}^{endeffo} = 0 \quad (2)$$
$$\mathbf{r}^{op_3} + \mathbf{r}^{p_3 r_3} + \mathbf{r}^{r_3 s_3} + \mathbf{r}^{s_3 endeff} + \mathbf{r}^{endeffo} = 0 \quad (3)$$

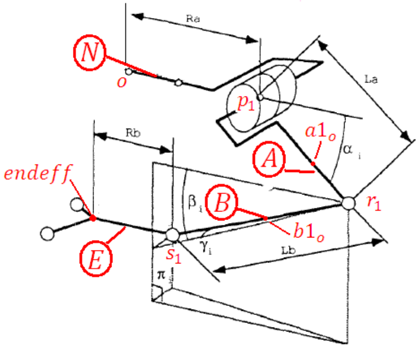


Figure: Delta robot.

$$\frac{\partial \mathcal{L}}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}, t) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}, t)\right) = Q_i, \tag{4}$$

where Q_i is the generalized force on the ith degree-of-freedom. The generalized forces (torques) can be calculated using the principle of virtual work

$$Q_i = \sum_{i=1}^{n} F_i \cdot \frac{\partial r_i}{\partial q_i}. \tag{5}$$

Further, F_i is the external force on the ith degree-of-freedom at point r_i.

Thanks

Questions please

1 Lagrange equations - Example 1

The planar mechanical system considered is shown in Fig. 1.1. has a slider 1 of mass m and a pendulum 2 with the mass M concentrated at B. The length of AB is L and the elastic constant of the spring R is k. The spring deflect only horizontally.

Given the initial conditions find the equations of motion using Lagrange method.

The generalized coordinates for this two degree of freedom system are the displacement of the slider $q_1(t)$ and the rotation of the pendulum $q_2(t)$

Position analysis

A cartesian reference frame $xOyz$ with the versors $[\mathbf{\imath},\ \mathbf{\jmath},\ \mathbf{k}]$ is selected, Fig. 1.1. The position vector of mass 1 is

$$\mathbf{r}_1 = \mathbf{r}_A = q_1(t)\,\mathbf{\imath}. \tag{1}$$

The position vector of mass 2 is

$$\mathbf{r}_2 = \mathbf{r}_B = [q_1(t) + L\sin q_2(t)]\,\mathbf{\imath} + L\cos q_2(t)\mathbf{\jmath}. \tag{2}$$

Velocity analysis

The velocity of the slider 1 is

$$\mathbf{v}_A = \frac{d\mathbf{r}_A}{dt} = \dot{\mathbf{r}}_A = \dot{q}_1\,\mathbf{\imath}, \tag{3}$$

and the velocity of the particle at B is

$$\mathbf{v}_B = \frac{d\mathbf{r}_B}{dt} = \dot{\mathbf{r}}_B = (\dot{q}_1 + L\dot{q}_2\cos q_2)\,\mathbf{\imath} - L\dot{q}_2\sin q_2\mathbf{\jmath}. \tag{4}$$

Kinetic energy

The kinetic energy of the slider 1 is

$$T_1 = \frac{1}{2}m\mathbf{v}_A\cdot\mathbf{v}_A = \frac{1}{2}m\dot{q}_1{}^2, \tag{5}$$

and the the kinetic energy of the mass 2 is

$$T_2 = \frac{1}{2}M\mathbf{v}_B\cdot\mathbf{v}_B = \frac{1}{2}M\left(\dot{q}_1{}^2 + 2L\dot{q}_1\dot{q}_2\cos q_2 + L^2\dot{q}_2{}^2\right). \tag{6}$$

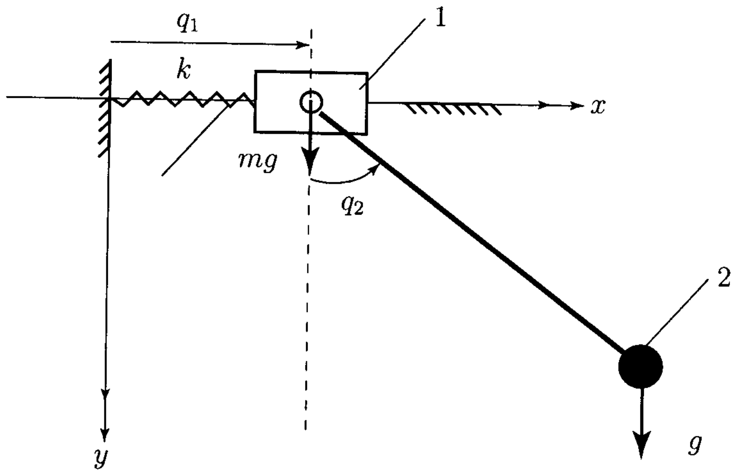


Figure 1

The total kinetic energy is

$$T = T_1 + T_2. \tag{7}$$

External forces analysis

The forces that act on 1 at A are the spring force and the gravity force

$$\mathbf{F}_A = -kq_1\,\imath + mg\,\jmath, \tag{8}$$

where g=9.81 m/s^2 is the gravity acceleration. The gravity force acts on mass 2 at B

$$\mathbf{F}_B = Mg\,\jmath. \tag{9}$$

Generalized forces

There are two generalized forces. The generalized force associated to q_1 is

$$\begin{aligned} &Q_1 = \mathbf{F}_A \cdot \frac{\partial \mathbf{r}_A}{\partial q_1} + \mathbf{F}_B \cdot \frac{\partial \mathbf{r}_B}{\partial q_1} = \\ &(-kq_1\,\imath + mg\,\jmath) \cdot \imath + Mg\,\jmath \cdot \imath = -kq_1. \end{aligned} \tag{10}$$

The generalized force associated to q_2 is

$$\begin{aligned} &Q_2 = \mathbf{F}_A \cdot \frac{\partial \mathbf{r}_A}{\partial q_2} + \mathbf{F}_B \cdot \frac{\partial \mathbf{r}_B}{\partial q_2} = \\ &(-kq_1\,\imath + mg\,\jmath) \cdot \mathbf{0} + Mg\,\jmath \cdot (L\cos q_2\,\imath - L\sin q_2\,\jmath) = \\ &-MgL\sin q_2. \end{aligned} \tag{11}$$

Lagrange equations

The two Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} &= Q_1, \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} &= Q_2. \end{aligned} \tag{12}$$

One can calculate for q_1

$$\begin{aligned} &\frac{\partial T}{\partial \dot{q}_1} = (m+M)\dot{q}_1 + LM\dot{q}_2\cos q_2, \\ &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = (m+M)\ddot{q}_1 + LM\ddot{q}_2\cos q_2 - LM\dot{q}_2{}^2\sin q_2, \\ &\frac{\partial T}{\partial q_1} = 0. \end{aligned} \tag{13}$$

For the generalized coordinate q_2 the left hand side of Lagrange equation is

$$\frac{\partial T}{\partial \dot{q}_2} = LM\left(\dot{q}_1 \cos q_2 + L\dot{q}_2\right),$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = LM\left(\ddot{q}_1 \cos q_2 - \dot{q}_1\dot{q}_2 \sin q_2 + L\ddot{q}_2\right),$$
$$\frac{\partial T}{\partial q_2} = -LM\dot{q}_1\dot{q}_2 \sin q_2. \tag{14}$$

The equations of motion are

$$(m+M)\ddot{q}_1 + LM\ddot{q}_2 \cos q_2 - LM\dot{q}_2{}^2 \sin q_2 = -kq_1,$$
$$LM\left(\ddot{q}_1 \cos q_2 - \dot{q}_1\dot{q}_2 \sin q_2 + L\ddot{q}_2\right) + LM\dot{q}_1\dot{q}_2 \sin q_2 = -MgL \sin q_2. \tag{15}$$

or

$$(m+M)\ddot{q}_1 + LM\ddot{q}_2 \cos q_2 - LM\dot{q}_2{}^2 \sin q_2 + kq_1 = 0,$$
$$LM\ddot{q}_1 \cos q_2 + ML^2\ddot{q}_2 + MgL \sin q_2 = 0. \tag{16}$$

For small oscillations of the pendulum $\sin q_2 \approx q_2$ and $\cos q_2 \approx 1$, the equations of motion are

$$(m+M)\ddot{q}_1 + LM\ddot{q}_2 - LM\dot{q}_2{}^2 q_2 + kq_1 = 0,$$
$$LM\ddot{q}_1 + ML^2\ddot{q}_2 + MgLq_2 = 0. \tag{17}$$

1 Lagrange equations - Example 2

A double pendulum is considered in Fig. 1.1. The bars 1 and 2 are homogenuos and have the lengths $OA = AB = L$ and the masses $m_1 = m_2 = m$. At O and A there are pin joints. The mass centers of links 1 and 2 are C_1 and C_2.

Find the Lagrange equations of motion if the initial conditions are known.

Link 1 can be rotated at O in a "fixed" cartesian reference frame of unit vectors $[\mathbf{\imath}, \mathbf{\jmath}, \mathbf{k}]$ about an axis $\mathbf{k}$. To characterize the instantaneous configuration of the system, two generalized coordinates $q_1(t)$ and $q_2(t)$ are employed. The generalized coordinates are quantities associated with the position of the system. The first generalized coordinate q_1 denotes the radian measure of the angle between the link 1 and the "fixed" cartesian reference frame. The last generalized coordinate q_2 designates also a radian measure of rotation angle between link 1 and link 2.

Kinematic analysis

The position vector of the mass center of link 1 is

$$\mathbf{r}_{C1} = 0.5\,L\,\sin q_1\,\mathbf{\imath} + 0.5\,L\,\cos q_1\,\mathbf{\jmath}, \tag{1}$$

the position vector of the mass center of link 2 is

$$\mathbf{r}_{C2} = (L\,\sin q_1 + 0.5\,L\,\sin q_2)\,\mathbf{\imath} + (L\,\cos q_1 + 0.5\,L\,\cos q_2)\,\mathbf{\imath}. \tag{2}$$

The velocity of C_1 is

$$\mathbf{v}_{C1} = \frac{d\,\mathbf{r}_{C1}}{d\,t} = \dot{\mathbf{r}}_{C1} = 0.5\,L\,\dot{q}_1\,\cos q_1\,\mathbf{\imath} - 0.5\,L\,\dot{q}_1\,\sin q_1\,\mathbf{\jmath}, \tag{3}$$

and the velocity of C_2 is

$$\mathbf{v}_{C2} = \frac{d\,\mathbf{r}_{C2}}{d\,t} = \dot{\mathbf{r}}_{C2} =$$
$$(L\,\dot{q}_1\,\cos q_1 + 0.5\,L\,\dot{q}_2\,\cos q_2)\,\mathbf{\imath} - (L\,\dot{q}_1\,\sin q_1 + 0.5\,L\,\dot{q}_2\,\sin q_2)\,\mathbf{\jmath}. \tag{4}$$

Kinetic energy

The kinetic energy of the link 1 which is in rotational motion is

$$T_1 = \frac{1}{2}\,I_0\,\dot{q}_1^2 = \frac{1}{2}\,\frac{ML^2}{3}\,\dot{q}_1^2 = \frac{ML^2}{6}\,\dot{q}_1^2, \tag{5}$$

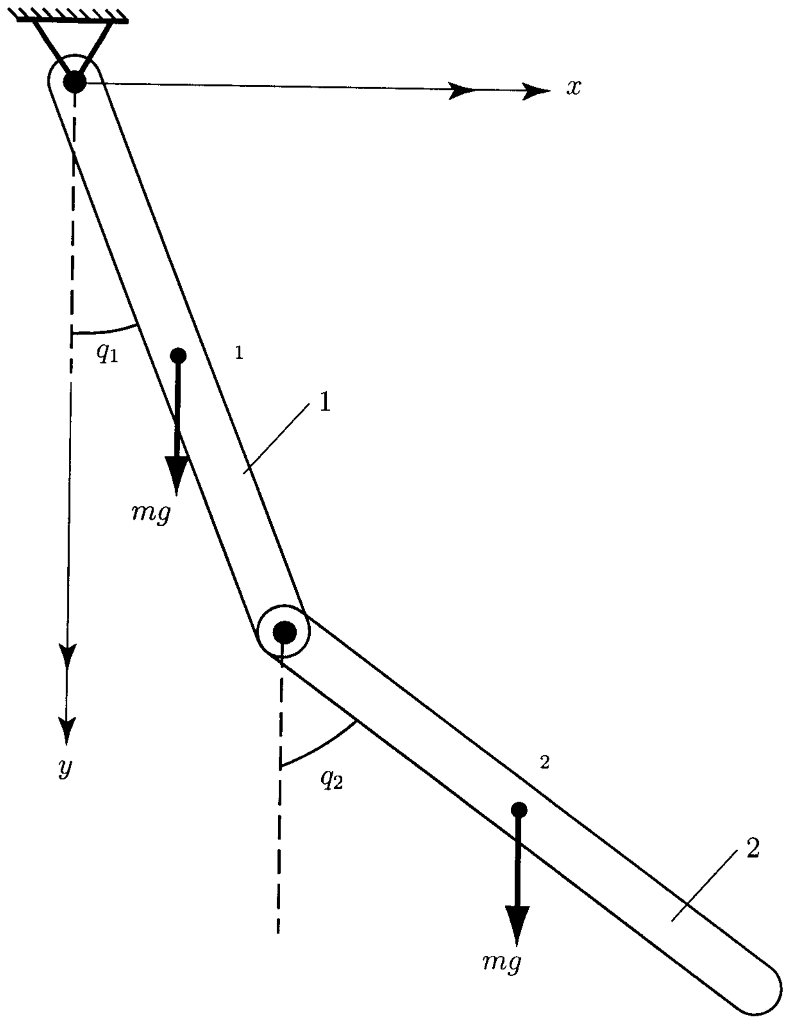


Figure 1

where I_0 is the mass moment of inertia about the center of rotation O, $I_O = mL^2/3$.
The kinetic energy of the bar 2 is due to the translation and rotation and can be expressed as

$$T_2 = \frac{1}{2} I_{C2} \dot{q}_2^2 + \frac{1}{2} m_2 \mathbf{v}_{C2}^2, \tag{6}$$

where I_{C2} is the mass moment of inertia about the center of mass C_2, $I_{C2} = mL^2/12$, and

$$\mathbf{v}_{C2}^2 = \mathbf{v}_{C2} \cdot \mathbf{v}_{C2} = L^2\, \dot{q}_1^2 + \frac{1}{4}\, L^2\, \dot{q}_2^2 + L^2\, \dot{q}_1\, \dot{q}_2 \cos(q_2 - q_1). \tag{7}$$

Equation (6) becomes

$$T_2 = \frac{1}{2}\, \frac{mL^2}{12}\, \dot{q}_2^2 + \frac{1}{2} mL^2 \left[\dot{q}_1^2 + \frac{1}{4}\, \dot{q}_2^2 + \dot{q}_1\, \dot{q}_2 \cos(q_2 - q_1)\right]. \tag{8}$$

The total kinetic energy of the system is

$$T = T_1 + T_2 = \frac{mL^2}{6} \left[4\dot{q}_1^2 + 3\, \dot{q}_1\, \dot{q}_2\, \cos(q_2 - q_1) + \dot{q}_2^2\right]. \tag{9}$$

The left hand sides of Lagrange equations are

$$\begin{aligned}
\frac{\partial T}{\partial \dot{q}_1} &= \frac{mL^2}{6}\, [8\dot{q}_1 + 3\dot{q}_2\, \cos(q_2 - q_1)]\,, \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) &= \frac{mL^2}{6}\, [8\ddot{q}_1 + 3\ddot{q}_2\, \cos(q_2 - q_1) - 3\dot{q}_2\, (\dot{q}_2 - \dot{q}_1)\, \sin(q_2 - q_1)]\,, \\
\frac{\partial T}{\partial q_1} &= \frac{mL^2}{6}\, 3\, \dot{q}_1\, \dot{q}_2\, \sin(q_2 - q_1) = \frac{mL^2}{2}\, \dot{q}_1\, \dot{q}_2\, \sin(q_2 - q_1); \\
\frac{\partial T}{\partial \dot{q}_2} &= \frac{mL^2}{6}\, [3\dot{q}_1\, \cos(q_2 - q_1) + 2\dot{q}_2]\,, \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) &= \frac{mL^2}{6}\, [3\ddot{q}_1\, \cos(q_2 - q_1) - 3\dot{q}_1\, (\dot{q}_2 - \dot{q}_1)\, \sin(q_2 - q_1) + 2\ddot{q}_2]\,, \\
\frac{\partial T}{\partial q_2} &= -\frac{mL^2}{6}\, 3\, \dot{q}_1\, \dot{q}_2\, \sin(q_2 - q_1) = -\frac{mL^2}{2}\, \dot{q}_1\, \dot{q}_2\, \sin(q_2 - q_1).
\end{aligned} \tag{10}$$

External forces analysis

The gravity forces on links 1 and 2 at the mass centers C_1 and C_2

$$\mathbf{F}_{C1} = \mathbf{F}_{C2} = mg\,\mathbf{J}. \tag{11}$$

Generalized forces

There are two generalized forces. The generalized force associated to q_1 is

$$\begin{aligned} Q_1 = \mathbf{F}_{C1} \cdot \frac{\partial \mathbf{r}_{C1}}{\partial q_1} + \mathbf{F}_{C2} \cdot \frac{\partial \mathbf{r}_{C2}}{\partial q_1} = \\ mg\,\mathbf{J} \cdot (0.5\,L\,\cos q_1\,\mathbf{I} - 0.5\,L\,\sin q_1\,\mathbf{J}) + mg\,\mathbf{J} \cdot (L\,\cos q_1\,\mathbf{I} - L\,\sin q_1\,\mathbf{J}) \\ = -1.5mgL\,\sin q_1. \end{aligned} \tag{12}$$

The generalized force associated to q_2 is

$$\begin{aligned} Q_2 = \mathbf{F}_{C1} \cdot \frac{\partial \mathbf{r}_{C1}}{\partial q_2} + \mathbf{F}_{C2} \cdot \frac{\partial \mathbf{r}_{C2}}{\partial q_2} = \\ mg\,\mathbf{J} \cdot \mathbf{0} + mg\,\mathbf{J} \cdot (0.5\,L\,\cos q_2\,\mathbf{I} - 0.5\,L\,\sin q_2\,\mathbf{J}) \\ = -0.5mgL\,\sin q_2. \end{aligned} \tag{13}$$

The two Lagrange equations are

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1, \\ &1.333mL^2\ddot{q}_1 + 0.5mL^2\ddot{q}_2\cos(q_2 - q_1) - 0.5mL^2\dot{q}_2^2\sin(q_2 - q_1) \\ &+1.5mgL\sin q_1 = 0; \\ \\ &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = Q_2, \\ &0.5mL^2\ddot{q}_1\cos(q_2 - q_1) + 0.333mL^2\ddot{q}_2 + 0.5mL^2\dot{q}_1^2\sin(q_2 - q_1) \\ &+0.5mgL\sin q_2 = 0. \end{aligned} \tag{14}$$

1 Lagrange equations - Example 3

Figure 1.1(a) is a schematic representation of an open kinematic chain (robot arm) consisting of three links 1, 2, 3, and a rigid body RB. Link 1 can be rotated at A in a "fixed" cartesian reference frame (0) of unit vectors $[\mathbf{\imath}_0, \mathbf{\jmath}_0, \mathbf{k}_0]$ about a vertical axis $\mathbf{\imath}_0$. The unit vector $\mathbf{\imath}_0$ is fixed in link 1. Link 1 is connected to link 2 through pin joints B and B'. The link 2 rotates relative to 1 about an axis fixed in both 1 and 2, passing through B, and B'. The link 3 is connected to 2 by means of a slider joint 2'. The slider joint is rigidly attached to link 2. The last link 3 holds rigidly the rigid body RB. The mass centers of links 1, 2, 2' and 3 are C_1, $C_2 = C_{2'}$, and C_3, respectively. The mass center of RB is C_R. The mass if the link 1 is m_1, the masses of the bars 2 and 3 are m_2 and m_3, the mass of the slider 2' is $m_{2'}$ and the mass of RB is m_R. The length of 2 is l and the length of 3 is L.

Find the equations of motion for the robotic system.

Solution

A reference frame (1) of unit vectors $[\mathbf{\imath}_1, \mathbf{\jmath}_1, \mathbf{k}_1]$ is attached to body 1, with $\mathbf{\imath}_1 = \mathbf{\imath}_0$.

A reference frame (2) of unit vectors $[\mathbf{\imath}_2, \mathbf{\jmath}_2, \mathbf{k}_2]$ is attached to link 2, as it is shown in Fig. 1.1. The unit vector $\mathbf{\jmath}_2$ is parallel to the axis of link 2, BB', and $\mathbf{\jmath}_2 = \mathbf{\jmath}_1$. The unit vector $\mathbf{k}_2$ is parallel to the axis of link 3, C_2C_R.

To characterize the instantaneous configuration of the arm, the *generalized coordinates* $q_1(t)$, $q_2(t)$, $q_3(t)$ are employed. The generalized coordinates are quantities associated with the position of the system.

The first generalized coordinate q_1 denotes the radian measure of the angle between the axes of (1) and (0) , Fig. 1.1(b). The second generalized coordinate q_2 designates also a radian measure of rotation angle between (1) and (2), Fig. 1.1(c). The last generalized coordinate q_3 is the distance from C_2 to C_3.

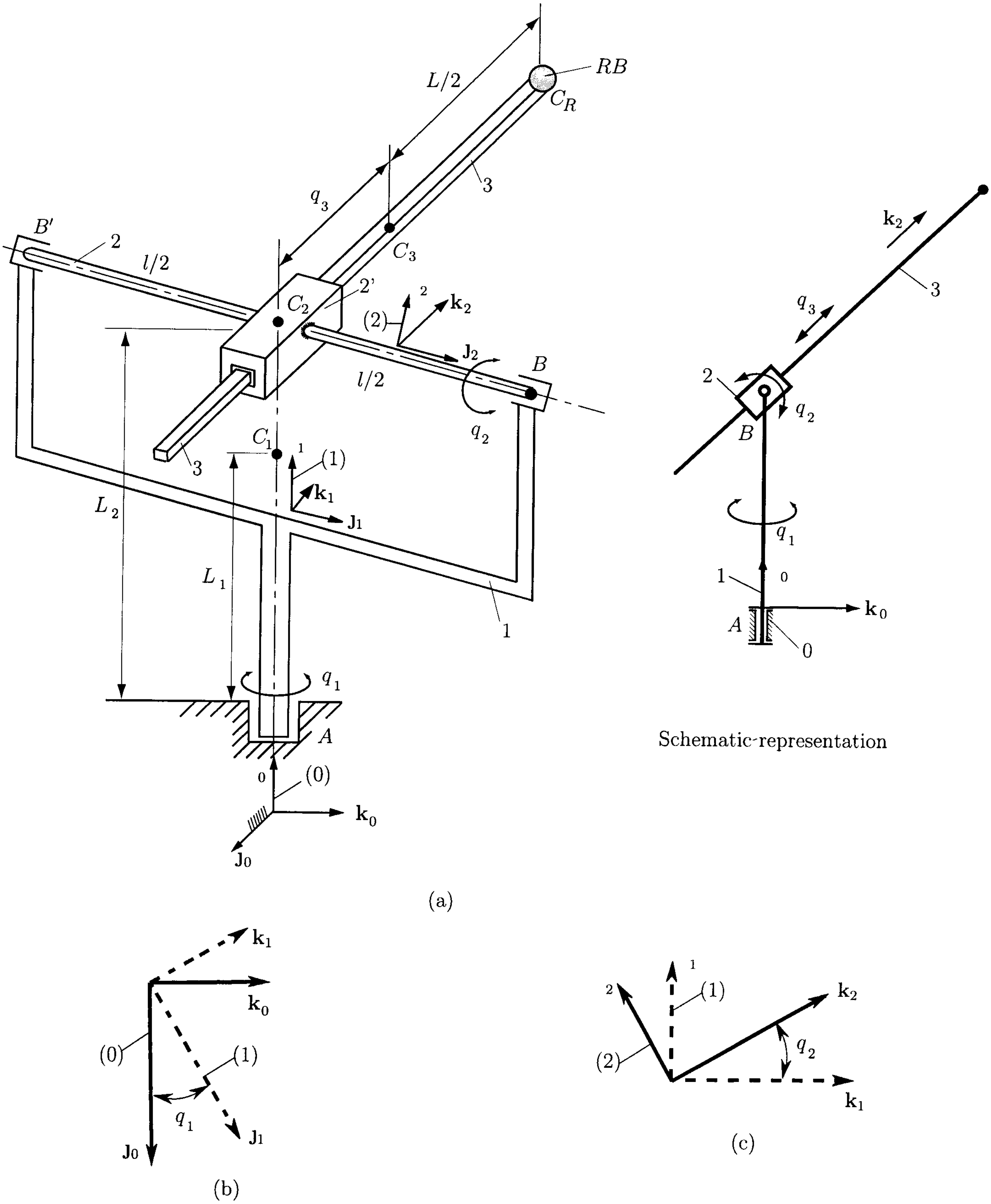
Angular velocities

Next the angular velocities of the links and the rigid body will be expressed in the fixed reference frame (0). One can express the angular velocity of link 1 in (0) as

$$\boldsymbol{\omega}_{10} = \dot{q}_1\, \mathbf{\imath}_1 = \dot{q}_1\, \mathbf{\imath}_0. \tag{1.1}$$

The angular velocity of link 2 with respect to (1) is

$$\boldsymbol{\omega}_{21} = \dot{q}_2\, \mathbf{\jmath}_2 = \dot{q}_2\, \mathbf{\jmath}_1, \tag{1.2}$$



L/2
RB
C_R
3
q_3
C_3
B′
2
l/2
C_2
2’
(2)
k_2
J_2
B
q_2
C_1
(1)
k_1
J_1
L_2
L_1
1
q_1
A
0
(0)
k_0
J_0
Schematic-representation
(a)
(b)
(c)

Figure 1

and the angular velocity of link 2 with respect to the fixed reference frame (0) is

$$\boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{21} = \dot{q}_1\, \mathbf{\imath}_1 + \dot{q}_2 \mathbf{\jmath}_2. \tag{1.3}$$

The unit vector $\mathbf{\imath}_1$, $\mathbf{\jmath}_1$ and $\mathbf{k}_1$ can be expressed as, Fig. 1.1(b)

$$\begin{aligned}
\mathbf{\imath}_1 &= \mathbf{\imath}_0, \\
\mathbf{\jmath}_1 &= c_1\, \mathbf{\jmath}_0 + s_1\, \mathbf{k}_0, \\
\mathbf{k}_1 &= -s_1\, \mathbf{\jmath}_0 + c_1\, \mathbf{k}_0,
\end{aligned} \tag{1.4}$$

where $s_1 = \sin q_1$ and $c_1 = \cos q_1$.

The unit vector $\mathbf{\imath}_2$, $\mathbf{\jmath}_2$ and $\mathbf{k}_2$ can be expressed as, Fig. 1.1(c)

$$\begin{aligned}
\mathbf{\imath}_2 &= c_2 \mathbf{\imath}_1 - s_2 \mathbf{k}_1 \\
&= c_2 \mathbf{\imath}_0 + s_1 s_2\, \mathbf{\jmath}_0 - c_1 s_2\, \mathbf{k}_0, \\
\mathbf{\jmath}_2 &= \mathbf{\jmath}_1, \\
&= c_1\, \mathbf{\jmath}_0 + s_1\, \mathbf{k}_0, \\
\mathbf{k}_2 &= s_2 \mathbf{\imath}_1 + c_2 \mathbf{k}_1 \\
&= s_2 \mathbf{\imath}_0 - c_2 s_1\, \mathbf{\jmath}_0 + c_1 c_2\, \mathbf{k}_0,
\end{aligned} \tag{1.5}$$

where $s_2 = \sin q_2$ and $c_2 = \cos q_2$.

The angular velocity of link 2 in (0) can be written in terms of the unit vectors of the reference frame (2) as

$$\boldsymbol{\omega}_{20} = \dot{q}_1 c_2 \mathbf{\imath}_2 + \dot{q}_2\, \mathbf{\jmath}_2 + \dot{q}_1 s_2\, \mathbf{k}_2. \tag{1.6}$$

The angular velocity of link 2 in (0) can be written in terms of the unit vectors of the reference frame (0) as

$$\boldsymbol{\omega}_{20} = \dot{q}_1 \mathbf{\imath}_0 + \dot{q}_2 c_1\, \mathbf{\jmath}_0 + \dot{q}_2 s_1\, \mathbf{k}_0. \tag{1.7}$$

The link 3 and the rigid body RB have the same rotation motion as link 2, i.e.

$$\boldsymbol{\omega}_{30} = \boldsymbol{\omega}_{R0} = \boldsymbol{\omega}_{20},$$

where $\boldsymbol{\omega}_{30}$ is the angular velocity of link 3 in (0) and $\boldsymbol{\omega}_{R0}$ is the angular velocity of RB in (0).

Linear velocities

The position vector of C_1, the mass center of link 1 is

$$\mathbf{r}_{C1} = L_1 \mathbf{\imath}_1 = L_1 \mathbf{\imath}_0, \tag{1.8}$$

and the velocity of C_1 in (0) is

$$\mathbf{v}_{C1} = \frac{d}{dt}\mathbf{r}_{C1} = \dot{\mathbf{r}}_{C1} = \mathbf{0}. \tag{1.9}$$

The position vector of C_2, the mass center of link 2, is

$$\mathbf{r}_{C2} = L_2\mathbf{1}_1 = L_2\mathbf{1}_0,$$

or written in terms of the unit vectors of the reference frame (2)

$$\mathbf{r}_{C2} = L_2c_2\mathbf{1}_2 + L_2s_2\mathbf{k}_2.$$

The velocity of C_2 in (0) is

$$\mathbf{v}_{C2} = \frac{d}{dt}\mathbf{r}_{C2} = \frac{d}{dt}(L_2\mathbf{1}_0) = \mathbf{0}.$$

The position vector of C_3 with respect to reference frame (0) is

$$\begin{aligned}\mathbf{r}_{C3} &= \mathbf{r}_{C2} + q_3\mathbf{k}_2 \\ &= L_2\mathbf{1}_0 + q_3\mathbf{k}_2,\end{aligned} \tag{1.10}$$

or expressing $\mathbf{k}_2$ in terms of reference (0) unit vectors yields

$$\mathbf{r}_{C3} = (L_2 + q_3s_2)\mathbf{1}_0 - q_3c_2s_1\,\mathbf{J}_0 + q_3c_2c_1\,\mathbf{k}_0.$$

The position vector of C_3 with respect to reference frame (0) written in terms of the unit vectors of the reference frame (2) is

$$\mathbf{r}_{C3} = L_2c_2\mathbf{1}_2 + (q_3 + L_2s_2)\mathbf{k}_2.$$

The velocity of the mass center C_3 in (0), written in terms of the unit vectors of the reference frame (0), can be calculated taking the derivative with respect to time of Eq. (1.11)

$$\begin{aligned}\mathbf{v}_{C3} = \frac{d}{dt}\mathbf{r}_{C3} &= (c_2q_3\dot{q}_2 + s_2\dot{q}_3)\mathbf{1}_0 + \\ &\quad (s_1s_2\dot{q}_2q_3 - c_1c_2q_3\dot{q}_1 - s_1c_2\dot{q}_3)\,\mathbf{J}_0 + \\ &\quad (c_1c_2\dot{q}_3 - s_1c_2q_3\dot{q}_1 - c_1s_2q_3\dot{q}_2)\,\mathbf{k}_0.\end{aligned} \tag{1.11}$$

The velocity of C_3 in (0) can be computed using the derivation formula for the moving vector $\mathbf{r}_{C3}$

$$\mathbf{v}_{C3} = \frac{d}{dt}\mathbf{r}_{C3} = \frac{{}^{(2)}\partial}{\partial t}\mathbf{r}_{C3} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{C3}, \qquad (1.12)$$

where $\frac{{}^{(2)}\partial}{\partial t}$ represents the partial derivative with respect to time in reference frame (2), $[\mathbf{\imath}_2, \mathbf{\jmath}_2, \mathbf{k}_2]$,

$$\frac{{}^{(2)}\partial}{\partial t}\mathbf{r}_{C3} = \frac{{}^{(2)}\partial}{\partial t}\left[L_2 c_2 \mathbf{\imath}_2 + (q_3 + L_2 s_2)\mathbf{k}_2\right] = -\dot{q}_2 L_2 s_2 \mathbf{\imath}_2 + (\dot{q}_3 + \dot{q}_2 L_2 c_2)\mathbf{k}_2 \qquad (1.13)$$

Using Eqs. (1.12)(1.13)(1.11)(1.7) the velocity of C_3 in (0), written in terms of the unit vectors of the reference frame (2) is

$$\begin{aligned} \mathbf{v}_{C3} &= -\dot{q}_2 L_2 s_2 \mathbf{\imath}_2 + (\dot{q}_3 + \dot{q}_2 L_2 c_2)\mathbf{k}_2 + \begin{vmatrix} \mathbf{\imath}_2 & \mathbf{\jmath}_2 & \mathbf{k}_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ L_2 c_2 & 0 & q_3 + L_2 s_2 \end{vmatrix} \\ &= \dot{q}_2 q_3 \mathbf{\imath}_2 - \dot{q}_1 q_3 c_2 \mathbf{\jmath}_2 + \dot{q}_3 \mathbf{k}. \end{aligned} \qquad (1.14)$$

The position vector of the mass center C_R of the rigid body RB is

$$\begin{aligned} \mathbf{r}_{CR} &= \mathbf{r}_{C3} + \mathbf{r}_{C3CR} \\ &= \mathbf{r}_{C3} + \frac{L}{2}\mathbf{k}_2, \end{aligned} \qquad (1.15)$$

or expressed in terms of the reference frame (0) is

$$\mathbf{r}_{CR} = \left[L_2 + \left(q_3 + \frac{L}{2}\right) s_2\right] \mathbf{\imath}_0 - \left(q_3 + \frac{L}{2}\right) c_2 s_1 \mathbf{\jmath}_0 + \left(q_3 + \frac{L}{2}\right) c_1 c_2 \mathbf{k}_0.$$

The velocity of C_R in (0) is

$$\begin{aligned} \mathbf{v}_{CR} &= \frac{d}{dt}\mathbf{r}_{CR} = \left[\left(q_3 + \frac{L}{2}\right) c_2 \dot{q}_2 + s_2 \dot{q}_3\right] \mathbf{\imath}_0 + \\ &\left[s_1 s_2 \dot{q}_2 \left(q_3 + \frac{L}{2}\right) - s_1 c_2 \dot{q}_3 - c_1 c_2 \dot{q}_1 \left(q_3 + \frac{L}{2}\right)\right] \mathbf{\jmath}_0 + \\ &\left[-c_2 s_1 \dot{q}_1 \left(q_3 + \frac{L}{2}\right) - c_1 s_2 \dot{q}_2 \left(q_3 + \frac{L}{2}\right) + c_1 c_2 \dot{q}_3\right] \mathbf{k}_0. \end{aligned} \qquad (1.16)$$

The velocity of C_R in (0) can be computed much easier using mobile reference frame (2)

$$\mathbf{v}_{CR} = \frac{d}{dt}\mathbf{r}_{CR} = \frac{^{(2)}\partial}{\partial t}\mathbf{r}_{CR} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{CR}, \tag{1.17}$$

where

$$\mathbf{r}_{CR} = L_2 c_2 \mathbf{\imath}_2 + (q_3 + L_2 s_2 + L/2)\mathbf{k}_2.$$

The velocity of C_R is

$$\begin{aligned} \mathbf{v}_{CR} &= -\dot{q}_2 L_2 s_2 \mathbf{\imath}_2 + (\dot{q}_3 + \dot{q}_2 L_2 c_2)\mathbf{k}_2 + \begin{vmatrix} \mathbf{\imath}_2 & \mathbf{\jmath}_2 & \mathbf{k}_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ L_2 c_2 & 0 & q_3 + L_2 s_2 + L/2 \end{vmatrix} \\ &= (L/2 + q_3)\dot{q}_2 \mathbf{\imath}_2 - c_2 \dot{q}_1 (q_3 + L/2)\mathbf{\jmath}_2 + \dot{q}_3 \mathbf{k}_2. \end{aligned} \tag{1.18}$$

Kinetic energy

The kinetic energy of a rigid body is

$$T = \frac{1}{2} m \mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2}\boldsymbol{\omega} \cdot (\bar{I} \cdot \boldsymbol{\omega}), \tag{1.19}$$

where m is the mass, $\mathbf{v}_C$ is the velocity of the mass center, $\boldsymbol{\omega} = \omega_x \mathbf{\imath} + \omega_y \mathbf{\jmath} + \omega_z \mathbf{k}$ is the angular velocity of the rigid body in (0), and $\bar{I} = (I_x \mathbf{\imath})\mathbf{\imath} + (I_y \mathbf{\jmath})\mathbf{\jmath} + (I_z \mathbf{k})\mathbf{k}$ is the central inertia *dyadic* of the rigid body. The central principal axes of the rigid body are parallel to $\mathbf{\imath}$, $\mathbf{\jmath}$, $\mathbf{k}$ and the associated moments of inertia have the values I_x, I_y, I_z, respectively. The inertia matrix associated to $\bar{I}$ is

$$\bar{I} \rightarrow \mathbf{I} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}. \tag{1.20}$$

The dot product of the vector $\boldsymbol{\omega}$ with the central inertia dyadic $\bar{I}$ is

$$\boldsymbol{\omega} \cdot \bar{I} = \bar{I} \cdot \boldsymbol{\omega} = \omega_x I_x \mathbf{\imath} + \omega_y I_y \mathbf{\jmath} + \omega_z I_z \mathbf{k}, \tag{1.21}$$

The total kinetic energy of the robot arm is

$$T = T_1 + T_2 + T_{2'} + T_3 + T_R, \tag{1.22}$$

where T_1 is the kinetic energy of link 1, T_2 is the kinetic energy of bar 2, $T_{2'}$ is the kinetic energy of slider 2', T_3 is the kinetic energy of bar 3, and T_R is the kinetic energy of RB.

The kinetic energy of link 1 is

$$T_1 = \frac{1}{2} m_1 \mathbf{v}_{C1} \cdot \mathbf{v}_{C1} + \frac{1}{2} \boldsymbol{\omega}_{10} \cdot (\bar{I}_1 \cdot \boldsymbol{\omega}_{10}) = \frac{1}{2} \boldsymbol{\omega}_{10} \cdot (\bar{I}_1 \cdot \boldsymbol{\omega}_{10}), \tag{1.23}$$

where m_1 is the mass of the link, $\bar{I}_1 = (I_{1x}\mathbf{\imath}_1)\mathbf{\imath}_1 + (I_{1y}\mathbf{\jmath}_1)\mathbf{\jmath}_1 + (I_{1z}\mathbf{k}_1)\mathbf{k}_1$ is the central inertia dyadic of link 1, and $\boldsymbol{\omega}_{10} = \dot{q}_1\,\mathbf{\imath}_1$. Using the above relation the kinetic energy of link 1 is

$$T_1 = \frac{1}{2} I_{1x} \dot{q}_1^2. \tag{1.24}$$

The kinetic energy of bar 2 is

$$T_2 = \frac{1}{2} m_2 \mathbf{v}_{C2} \cdot \mathbf{v}_{C2} + \frac{1}{2} \boldsymbol{\omega}_{20} \cdot (\bar{I}_2 \cdot \boldsymbol{\omega}_{20}) = \frac{1}{2} \boldsymbol{\omega}_{20} \cdot (\bar{I}_2 \cdot \boldsymbol{\omega}_{20}), \tag{1.25}$$

where m_2 is the mass of the bar and

$$\bar{I}_2 = (I_{2x}\mathbf{\imath}_2)\mathbf{\imath}_2 + (I_{2y}\mathbf{\jmath}_2)\mathbf{\jmath}_2 + (I_{2z}\mathbf{k}_1)\mathbf{k}_1 = (\frac{m_2 l^2}{12}\mathbf{\imath}_2)\mathbf{\imath}_2 + (\frac{m_2 l^2}{12}\mathbf{k}_2)\mathbf{k}_2,$$

is the central inertia dyadic of bar 2 with the length l. The kinetic energy is

$$T_2 = \frac{m_2 l^2}{24} \dot{q}_1^2. \tag{1.26}$$

The kinetic energy of slider 2' is

$$T_{2'} = \frac{1}{2} m_{2'} \mathbf{v}_{C2'} \cdot \mathbf{v}_{C2'} + \frac{1}{2} \boldsymbol{\omega}_{20} \cdot (\bar{I}_{2'} \cdot \boldsymbol{\omega}_{20}) = \frac{1}{2} \boldsymbol{\omega}_{20} \cdot (\bar{I}_{2'} \cdot \boldsymbol{\omega}_{20}), \tag{1.27}$$

where $m_{2'}$ is the mass of the slider and

$$\bar{I}_{2'} = (I_{2'x}\mathbf{\imath}_2)\mathbf{\imath}_2 + (I_{2'y}\mathbf{\jmath}_2)\mathbf{\jmath}_2 + (I_{2'z}\mathbf{k}_2)\mathbf{k}_2,$$

is the central inertia dyadic of the slider. The kinetic energy is

$$T_{2'} = \frac{1}{2}\left[(I_{2'x}c_2^2 + I_{2'z}s_2^2)\dot{q}_1^2 + I_{2'y}\dot{q}_2^2\right]. \tag{1.28}$$

The kinetic energy of bar 3 is

$$T_3 = \frac{1}{2} m_3 \mathbf{v}_{C3} \cdot \mathbf{v}_{C3} + \frac{1}{2} \boldsymbol{\omega}_{20} \cdot (\bar{I}_3 \cdot \boldsymbol{\omega}_{20}), \tag{1.29}$$

where m_3 is the mass of the bar and

$$\bar{I}_3 = (I_{3x}\mathbf{1}_2)\mathbf{1}_2 + (I_{3y}\mathbf{J}_2)\mathbf{J}_2 + (I_{3z}\mathbf{k}_2)\mathbf{k}_2 = (\frac{m_3 L^2}{12}\mathbf{1}_2)\mathbf{1}_2 + (\frac{m_3 L^2}{12}\mathbf{J}_2)\mathbf{J}_2,$$

is the central inertia dyadic of bar 3.

The rigid body RB is considered a particle with the mass m_R concentrated at C_R. The kinetic energy of RB is

$$T_R = \frac{1}{2} m_R \mathbf{v}_{CR} \cdot \mathbf{v}_{CR} = \frac{m_R}{2}\left[(L/2+q_3)^2\dot{q}_2^2 + c_2^2(q_3+L/2)^2\dot{q}_1^2 + \dot{q}_3^2\right]. \quad (1.30)$$

Generalized forces

In the case of the robot arm, there are two kinds of forces that contribute to the generalized forces Q_1, Q_2, Q_3 namely, contact forces applied in order to drive 1, 2, 3 and RB, and gravitational forces exerted on 1, 2, 3, and RB by the Earth. The contact forces are neglected for this example. The gravitational forces exerted on 1, 2, 3, and RB by the Earth, are denoted by $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{G}_3$, $\mathbf{G}_R$, respectively, and can be expressed as

$$\begin{aligned}
\mathbf{G}_1 &= -m_1\, g\, \mathbf{1}_0, \\
\mathbf{G}_2 &= -(m_2 + m_{2'})\, g\, \mathbf{1}_0, \\
\mathbf{G}_3 &= -m_3\, g\, \mathbf{1}_0, \\
\mathbf{G}_R &= -m_R\, g\, \mathbf{1}_0. \qquad (1.31)
\end{aligned}$$

One can express the contribution to the generalized force of all forces and torques acting on the system, as

$$Q_r = \frac{\partial \mathbf{r}_{C1}}{\partial q_r}\cdot \mathbf{G}_1 + \frac{\partial \mathbf{r}_{C2}}{\partial q_r}\cdot \mathbf{G}_2 + \frac{\partial \mathbf{r}_{C3}}{\partial q_r}\cdot \mathbf{G}_3 + \frac{\partial \mathbf{r}_{CB}}{\partial q_r}\cdot \mathbf{G}_R, \quad r = 1,2,3. \quad (1.32)$$

The vectors in Eq. (1.32) must be expressed in terms of the fixed reference frame (0). The generalized forces are

$$\begin{aligned}
Q_1 &= 0, \\
Q_2 &= -g c_2 (m_r L/2 + m_r q_3 + m_3 q_3), \\
Q_3 &= -g(m_R + m_3) s_2.
\end{aligned}$$

The same results can be obtained using the relations

$$Q_r = \frac{\partial \mathbf{v}_{C1}}{\partial \dot{q}_r}\cdot \mathbf{G}_1 + \frac{\partial \mathbf{v}_{C2}}{\partial \dot{q}_r}\cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_{C3}}{\partial \dot{q}_r}\cdot \mathbf{G}_3 + \frac{\partial \mathbf{v}_{CB}}{\partial \dot{q}_r}\cdot \mathbf{G}_R, \quad r = 1,2,3. \quad (1.33)$$

In Eq. (1.33) the vectors $\mathbf{v}_{C1}$, $\mathbf{G}_1$ are expressed in terms of the mobile reference frame (1), and the vectors $\mathbf{v}_{C2}$, $\mathbf{G}_2$, $\mathbf{v}_{C3}$, $\mathbf{G}_3$, $\mathbf{v}_{CR}$, $\mathbf{G}_R$ are expressed in terms of the mobile reference frame (2).

The Lagrange equations of motion are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r,$$

where $r = 1, 2, 3$.