Robotics

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Fundamental concepts

In optimal control problems the objective is to determine a function that minimizes a specified functional, i.e., the performance measure. The analogous problem in calculus is to determine a point that yields the minimum value of a function.

Fundamental concepts

Calculus of variations deals with certain kinds of external problems in which expressions involving integrals are optimized (maximized or minimized). Euler and Lagrange in the 18th century laid the foundations, with the classical problems of determining a closed curve in the plane enclosing maximum area subject to fixed length and the brachistochrone problem of determining the path between two points in minimum time. The present day problems include the maximization of the entropy integral in third law of thermodynamics, minimization of potential and kinetic energies integral in Hamiltons principle in mechanics, the minimization of energy integral in the problems in elastic behaviour of beams, plates and shells. Thus calculus of variations deals with the study of extrema of functionals.

Functional

A real valued function f whose domain is the set of real functions $y(x)$ is known as a functional (or functional of a single independent variable). Thus the domain of definition of a functional is a set of admissible functions. In ordinary functions the values of the independent variables are numbers. Whereas with functionals, the values of the independent variables are functions.

Let us estimate the length of the curve. We will do this by dividing the interval up into n equal subintervals each of width and we will denote the point on the curve at each point by P_i. We can then approximate the curve by a series of straight lines connecting the points. Here is a sketch of this situation for $n = 9$:

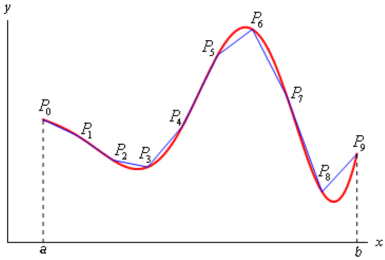


Figure: Length of a curve.

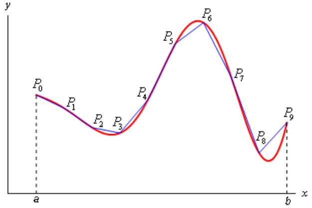
Functional

Now denote the length of each of these line segments by $\mid P_{i-1}P_i \mid$ and the length of the curve will then be approximately,

$$L \approx \sum_{i=1}^{n} \mid P_{i-1}P_i \mid, \tag{1}$$

and we can get the exact length by taking n larger and larger. In other words, the exact length will be,

$$L = \lim \sum_{i1=1}^{n} \mid P_{i-1}P_i \mid, \tag{2}$$



Let us get a better grasp on the length of each of these line segments. First, on each segment lets define $\Delta y_i = y_i - y_{i-1} = f(x_i) - f(xi - 1)$. We can then compute directly the length of the line segments as follows:

$$\mid P_{i-1}P_i \mid = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\Delta x_i^2 + \Delta y_i^2}. \quad (3)$$

By the Mean Value Theorem we know that on the interval $[x_{i-1}, x_i]$ there is a point x_i^* so that,

$$f(x_i) - f(x_{i-1}) = f^{'}(x_i^*)(x_i - x_{i-1}); \quad (4)$$

$$\Delta y_i = f^{'}(x_i^*)\Delta x_i. \quad (5)$$

Therefore, the length can now be written as,

$$\mid P_{i-1}P_i \mid = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}. \quad (6)$$

$$| P_{i-1}P_i |= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}, \quad (7)$$

$$| P_{i-1}P_i |= \sqrt{\Delta x^2 + [f'(x_i^*)]^2 \Delta x^2}, \quad (8)$$

$$| P_{i-1}P_i |= \sqrt{1 + [f'(x_i^*)]^2} \Delta x. \quad (9)$$

The exact length of the curve is then,

$$L = \lim \sum_{i1=1}^{n} | P_{i-1}P_i |, \quad (10)$$

$$L = \lim \sum_{i1=1}^{n} \sqrt{1 + [f'(x_i^*)]^2} \Delta x. \quad (11)$$

Using the definition of the definite integral, this is nothing more than,

$$L = \int_a^b \sqrt{1 + [f'(x_i^*)]^2 \mathrm{d}x}. \quad (12)$$

The length L of a curve, c whose equation is $y = f(x)$, passing through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$L = \int_{x_1}^{x_2} \sqrt{1 + [y']^2}\mathrm{d}x, \tag{13}$$

where $y^{'}$ denotes derivative of y with respect to x.
Now the length L of the curve passing through A and B depends on $y(x)$ (the curve). Then L is a function of the independent variable $y(x)$, which is a function. Thus

$$L(y(x)) = \int_{x_1}^{x_2} \sqrt{1 + [y']^2}\mathrm{d}x, \tag{14}$$

defines a functional which associates a real number L uniquely to each $y(x)$ (the independent variable). Further, suppose we wish to determine the curve having shortest (least) distance between the two given points A and B, i.e., curve with minimum length L.

This is a classical example of a variational problem in which we wish to determine, the particular curve $y = y(x)$ which minimizes the functional $Ly(x)$ given by (1). Here the two conditions $y(x_1) = y_1$ and $y(x_2) = y_2$, which are imposed on the curve $y(x)$ are known as end conditions of the problem. Thus variational problems involves determination of maximum or minimum or stationary values of a functional. The term extremum is used to include maximum or minimum or stationary values.

The Increment of a Functional

Definition

If $\mathbf{q}$ and $\mathbf{q} + \Delta\mathbf{q}$ are elements for which the function f is defined, then the increment of f, denoted by Δf is given by

$$\Delta f \triangleq f(\mathbf{q} + \Delta\mathbf{q}) - f(\mathbf{q}). \tag{15}$$

Consider the function

$$f(\mathbf{q}) = q_1^2 + 2q_1q_2. \tag{16}$$

The increment of f is

$$\begin{aligned} \Delta f &= [q_1 + \Delta q_1]^2 + 2(q_1 + \Delta q_1)(q_2 + \Delta q_2) - q_1^2 - 2q_1q_2 \\ &= 2q_1\Delta q_1 + [\Delta q_1]^2 + 2\Delta q_1 q_2 + 2\Delta q_2 q_1 + 2\Delta q_1 \Delta q_2. \end{aligned}$$

The Increment of a Functional

Definition

If $\mathbf{x}$ and $\mathbf{x} + \delta\mathbf{x}$ are functions for which the functional J is defined, then the increment of J, denoted by ΔJ is given by

$$\Delta J \triangleq J(\mathbf{x} + \delta\mathbf{x}) - J(\mathbf{x}). \tag{17}$$

Consider the functional

$$J(\mathbf{x}) = \int_{t_0}^{t_f} x^2(t)\mathrm{d}t. \tag{18}$$

The increment of J is

$$\begin{aligned} \Delta J &= \int_{t_0}^{t_f} [x(t) + \delta x(t)]^2 \mathrm{d}t - \int_{t_0}^{t_f} x^2(t)\mathrm{d}t \\ &= \int_{t_0}^{t_f} [2x(t)\delta x(t) + [\delta x(t)^2]\mathrm{d}t. \end{aligned}$$

The Variation of a Functional

Definition

The increment of a function of n variables can be written as

$$\Delta f(\mathbf{q}, \Delta\mathbf{q}) = \mathrm{d}f(\mathbf{q}, \Delta\mathbf{q}) + g(\mathbf{q}, \Delta\mathbf{q}) \parallel \Delta\mathbf{q} \parallel . \tag{19}$$

where $\mathrm{d}f$ is a linear function of $\Delta\mathbf{q}$ if

$$\lim_{\parallel\Delta\mathbf{q}\parallel \longrightarrow 0} \{g(\mathbf{q}, \Delta\mathbf{q})\} = 0 \tag{20}$$

then f is said to be differentiable at $\mathbf{q}$, and $\mathrm{d}f$ is the differential of f at the point $\mathbf{q}$.

The Variation of a Functional

Find the differential of

$$f(\mathbf{q}) = q_1^2 + 2q_1q_2. \tag{21}$$

We found the increment that is given by

$$\Delta f = 2q_1\Delta q_1 + 2\Delta q_1 q_2 + 2\Delta q_2 q_1 + [\Delta q_1]^2 + 2\Delta q_1\Delta q_2. \tag{22}$$

The first two terms are linear in Δq, letting

$$\parallel \Delta\mathbf{q} \parallel = \sqrt{[\Delta q_1]^2 + [\Delta q_2]^2}, \tag{23}$$

we can write the last two terms as

$$\frac{2\Delta q_1\Delta q_2}{\sqrt{[\Delta q_1]^2 + [\Delta q_2]^2}} \cdot \sqrt{[\Delta q_1]^2 + [\Delta q_2]^2}, \tag{24}$$

which is of the form $g(\mathbf{q}, \Delta\mathbf{q}) \parallel \Delta\mathbf{q} \parallel$. To show that f is differentiable we must verify that

$$\lim_{\parallel \Delta\mathbf{q} \parallel \longrightarrow 0} \frac{2\Delta q_1 \Delta q_2}{\sqrt{[\Delta q_1]^2 + [\Delta q_2]^2}} = 0. \tag{25}$$

Therefore, f is differentiable and the differential is given by

$$\mathrm{d}f(\mathbf{q}, \Delta\mathbf{q}) = [2q_1 + 2q_2]\Delta q_1 + [2q_1]\Delta q_2. \tag{26}$$

Definition

The increment of a functional J can be written as

$$\Delta J(\mathbf{x}, \delta\mathbf{x}) = \delta J(\mathbf{x}, \delta\mathbf{x}) + g(\mathbf{x}, \delta\mathbf{x}) \parallel \delta\mathbf{x} \parallel . \tag{27}$$

where δJ is a linear function of $\delta\mathbf{x}$ if

$$\lim_{\parallel\delta\mathbf{x}\parallel \longrightarrow 0} \{g(\mathbf{x}, \delta\mathbf{x})\} = 0 \tag{28}$$

then J is said to be differentiable on $\mathbf{x}$, and δJ is the variation of J evaluated for the function $\mathbf{x}$.

Find the variation of the functional J

$$J(\mathbf{x}) = \int_0^1 [x^2(t) + 2x(t)]\mathrm{d}t. \tag{29}$$

The increment of J is given by

$$\Delta J(x, \delta x) = \int_0^1 \{[x(t)+\delta x(t)]^2+2[x(t)+\delta x(t)]\}\mathrm{d}t-\int_0^1 [x^2(t)+2x(t)]\mathrm{d}t. \tag{30}$$

Expanding and combining these integrals, we obtain

$$\Delta J(x, \delta x) = \int_0^1 \{[2x(t) + 2]\delta x(t) + [\delta x(t)]^2\}\mathrm{d}t. \tag{31}$$

Separating the terms that are linear in δx, we obtain

$$\Delta J(x, \delta x) = \int_0^1 [2x(t) + 2]\delta x(t)\mathrm{d}t + \int_0^1 [\delta x(t)]^2\mathrm{d}t. \tag{32}$$

$$\Delta J(x, \delta x) = \int_0^1 [2x(t) + 2]\delta x(t)\mathrm{d}t + \int_0^1 [\delta x(t)]^2 \mathrm{d}t. \qquad (33)$$

Therefore, the variation of J is

$$\delta J(x, \delta x) = \int_0^1 [2x(t) + 2]\delta x(t)\mathrm{d}t. \qquad (34)$$

The Fundamental Theorem of the Calculus of Variation

Theorem (The fundamental theorem of the calculus of variation)

If x^* is an extermal, the variation of J must vanish on x^*, i.e., $\delta J(x^*, \delta x) = 0$ for all admissible δx.

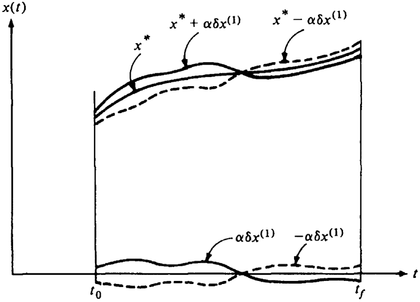


Figure: An extremal and two neighboring curves. Image courtesy of Donald E. Kirk.

- Euler equation is a necessary condition for x^* to be an extremal

$$\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right) = 0. \quad (35)$$

- Lagrange's equations

$$\frac{\partial \mathcal{L}}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}, t) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}, t)\right) = 0, \quad (36)$$

where $\mathcal{L}$ is the *Lagrangian* function. Further, q_i is the generalized coordinates of the ith degree-of-freedom, respectively.

The Fundamental Theorem of the Calculus of Variation

A necessary condition for the integral

$$J(x(t)) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\mathrm{d}t. \quad (37)$$

to attain an extreme value is that the extremizing function $x(t)$ should satisfy

$$\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right) = 0. \quad (38)$$

The Fundamental Theorem of the Calculus of Variation

Note 1

The second-order differential equations (35 and 36) is known as Euler-Lagrange or simply Eulers equation for the integral (1).

Note 2

The solutions (integral curves) of Eulers equation are known as extremals (or stationary functions) of the functional. Extremum for a functional can occur only on extremals.

Find the shortest smooth plane curve joining two distinct points in the plane.

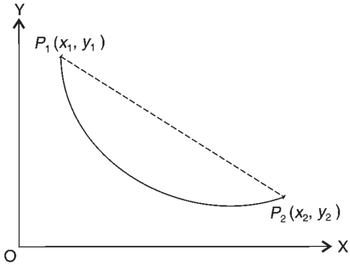


Figure: Shortest distance problem.

Assume that the two distinct points be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie in the XY-Plane. If $y = f(x)$ is the equation of any plane curve c in XY-Plane and passing through the points P_1 and P_2, then the length L of curve c is given by

$$L(y(x)) = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\mathrm{d}x \tag{39}$$

The variational problem is to find the plane curve whose length is shortest i.e., to determine the function $y(x)$ which minimizes the functional $L(y(x))$. The condition for extrema is the Eulers equation

$$\frac{\delta f}{\delta y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\delta f}{\delta y'}\right) = 0 \tag{40}$$

where f is given by

$$f = \sqrt{1+(y')^2}. \tag{41}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0, \tag{42}$$

Integration of the previous conditions yields, $y(x) = mx + c$, where m and c are constants of he integration.

Shortest Time

Determine the plane curve down which a particle will slide without friction from the point $A(x_1, y_1)$ to $B(x_2, y_2)$ in the shortest time.

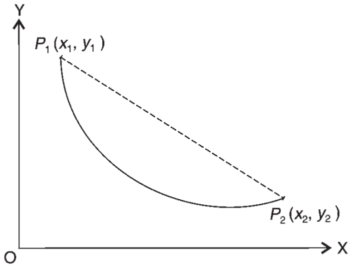


Figure: Shortest time problem.

Assume the positive direction of the y-axis is vertically downward and let $x_1 < x_2$. Let $P(x, y)$ be the position of the particle at any time t, on the curve c. Since energy is conserved, the speed v of the particle sliding along any curve is given by

Shortest Time

$$v = \sqrt{2g(y - y^*)} \tag{43}$$

where g is the acceleration due to gravity, and $y^* = y_1 - \left(\frac{v_1^2}{2g}\right)$. Further, v_1 is the initial speed. Choose the origin at A so that $x_1 = 0$, $y_1 = 0$ and assume that $v_1 = 0$. Then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v = \sqrt{2gy} \tag{44}$$

Integrating this, we get the time taken by the particle moving under gravity (and neglecting friction along the curve and neglecting resistance of the medium) from $A(0,0)$ to $B(x_2, y_2)$ is

$$t(y(x)) = \int \frac{\mathrm{d}s}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}} \int_0^{x_2} \frac{\sqrt{1 + y'^2}}{\sqrt{y}} \mathrm{d}x \tag{45}$$