Robotics

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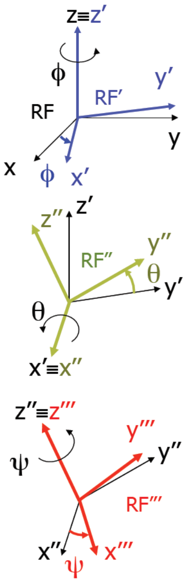
$ZX'Z''$ Euler angles



Rotation around Z-axis

$$\mathbf{R}_z(\Phi) = \begin{bmatrix} \cos\Phi & -\sin\Phi & 0 \\ \sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation around X'-axis

$$\mathbf{R}_{x'}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Rotation around Z''-axis

$$\mathbf{R}_{z''}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$ZX'Z''$ Euler angles

Direct problem

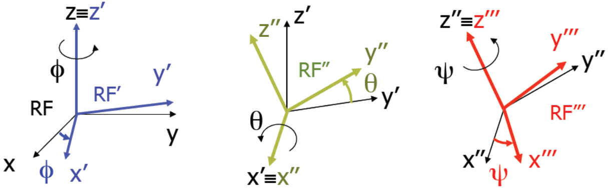
Given Φ, θ, and ψ. Find $\mathbf{R}$

$$\mathbf{R}_{zx'z''} = \mathbf{R}_z(\Phi)\mathbf{R}_{x'}(\theta)\mathbf{R}_{z''}(\psi) \quad (1)$$

$$= \begin{bmatrix} \cos\Phi\cos\psi - \sin\Phi\cos\theta\sin\psi & -\cos\Phi\sin\psi - \sin\Phi\cos\theta\cos\psi & \sin\Phi\sin\theta \\ \sin\Phi\cos\psi + \cos\Phi\cos\theta\sin\psi & -\sin\Phi\sin\psi + \cos\Phi\cos\theta\cos\psi & -\cos\Phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}$$

- Given a vector $\mathbf{v}''' = (x''', y''', z''')$ expressed in RF"', its expression in coordinates of RF is

$$\mathbf{v} = \mathbf{R}_{zx'z''}\mathbf{v}''' \quad (2)$$



Inverse problem

Given $\mathbf{R}$. Find Φ, θ, and ψ

$$\mathbf{R}_{zx'z''} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad (3)$$

$$= \begin{bmatrix} \cos\Phi\cos\psi - \sin\Phi\cos\theta\sin\psi & -\cos\Phi\sin\psi - \sin\Phi\cos\theta\cos\psi & \sin\Phi\sin\theta \\ \sin\Phi\cos\psi + \cos\Phi\cos\theta\sin\psi & -\sin\Phi\sin\psi + \cos\Phi\cos\theta\cos\psi & -\cos\Phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}$$

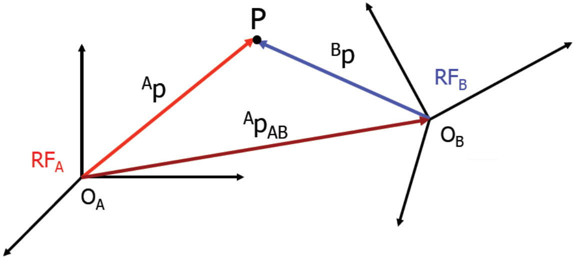
- $r_{13}^2 + r_{23}^2 = \sin^2\theta \quad , \quad r_{33} = \cos\theta$
  $\Rightarrow \quad \theta = \mathrm{atan2}\left(\pm\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$
- If $r_{13}^2 + r_{23}^2 \neq 0$ $(\sin\theta \neq 0)$ $r_{31}/\sin\theta = \sin\psi$, $r_{32}/\sin\theta = \cos\psi$
  $\Rightarrow \quad \psi = \mathrm{atan2}\left(r_{31}/\sin\theta, r_{32}/\sin\theta\right)$
- Finally, $\Rightarrow \quad \Phi = \mathrm{atan2}\left(r_{13}/\sin\theta, -r_{23}/\sin\theta\right)$

Homogeneous transformations

$$^{A}\mathbf{p} = \, ^{A}\mathbf{p}_{AB} + \, ^{A}\mathbf{R}^{B} \, ^{B}\mathbf{p} \qquad \textbf{Affine Relationship} \qquad (4)$$

Vector in homogeneous coordinates (Linear Relationship)

$$^{A}\mathbf{p}_{\text{hom}} = \begin{bmatrix} ^{A}\mathbf{p} \\ -- \\ 1 \end{bmatrix} \begin{bmatrix} ^{A}\mathbf{R}^{B} & | & ^{A}\mathbf{p}_{AB} \\ -- & -- & -- \\ 0 \;\; 0 \;\; 0 & | & 1 \end{bmatrix} \begin{bmatrix} ^{B}\mathbf{p} \\ -- \\ 1 \end{bmatrix} = \, ^{A}\mathbf{T}^{B} \, ^{B}\mathbf{p}_{\text{hom}}$$



Homogeneous transformations

Vector in homogeneous coordinates (Linear Relationship)

$$^{A}\mathbf{p}_{\text{hom}} = \begin{bmatrix} ^{A}\mathbf{p} \\ -- \\ 1 \end{bmatrix} \begin{bmatrix} & ^{A}\mathbf{R}^{B} & & | & ^{A}\mathbf{p}_{AB} \\ & -- & & -- & -- \\ 0 & 0 & 0 & | & 1 \end{bmatrix} \begin{bmatrix} ^{B}\mathbf{p} \\ -- \\ 1 \end{bmatrix} = \; ^{A}\mathbf{T}^{B} \; ^{B}\mathbf{p}_{\text{hom}}$$

- describes the relation between reference frames (relative pose = position and orientation)
- transforms the representation of a position vector (applied vector starting from the origin of the frame) from a given frame to another frame
- it is a roto-translation operator on vectors in the three-dimensional space
- it is always invertible $(^{A}\mathbf{T}^{B})^{-1} = \; ^{B}\mathbf{T}^{A}$
- can be composed $^{A}\mathbf{T}^{C} = \; ^{A}\mathbf{T}^{B} \; ^{B}\mathbf{T}^{C}$

Homogeneous transformations

Translations

$$ {}^{A}\mathbf{T}^{B} = \begin{bmatrix} \mathbf{I}_{3\times3} & {}^{A}\mathbf{p}_{AB} \\ 0_{1\times3} & 1 \end{bmatrix} $$

Rotations

$$ {}^{A}\mathbf{T}^{B} = \begin{bmatrix} {}^{A}\mathbf{R}^{B} & 0_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix} $$

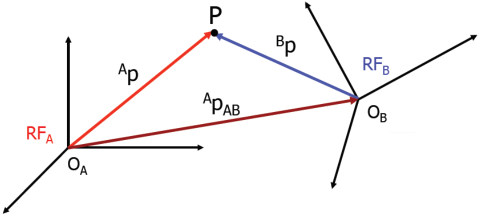


Figure: Homogeneous transformations.

Homogeneous transformations

$$ {}^{\mathrm{W}}\mathbf{T}^{\mathrm{T}} = {}^{\mathrm{W}}\mathbf{T}^{\mathrm{B}}\ {}^{\mathrm{B}}\mathbf{T}^{\mathrm{E}}\ {}^{\mathrm{E}}\mathbf{T}^{\mathrm{T}} $$

- ${}^{\mathrm{W}}\mathbf{T}^{\mathrm{T}}$: absolute definition of task.
- ${}^{\mathrm{W}}\mathbf{T}^{\mathrm{B}}$: is based on the initial position of the robot.
- ${}^{\mathrm{B}}\mathbf{T}^{\mathrm{E}}$: direct kinematics of the robot arm ($\varphi(\mathbf{q})$).
- ${}^{\mathrm{E}}\mathbf{T}^{\mathrm{T}}$: task definition relative to the robot end-effector

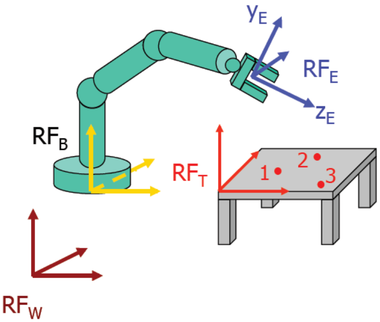


Figure: robot task

$$ {}^{\mathrm{B}}\mathbf{T}^{\mathrm{E}} = ({}^{\mathrm{W}}\mathbf{T}^{\mathrm{B}})^{-1}\ {}^{\mathrm{W}}\mathbf{T}^{\mathrm{T}}\ ({}^{\mathrm{E}}\mathbf{T}^{\mathrm{T}})^{-1} $$

Homogeneous transformations

Find the homogeneous transformation matrix ($\mathbf{T}$) for the following operations:

- Rotation α around x-axis
- Translation with a along x-axis
- Translation with d along z-axis
- Rotation θ around z-axis

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous transformations

You are have a three link arm that starts out aligned in the x-axis. Each link has lengths l_1, l_2, l_3, respectively. You tell the first one to move by q_1 , and so on as the diagram suggests. Find the Homogeneous matrix to get the position of the yellow dot in the X^0Y^0 frame

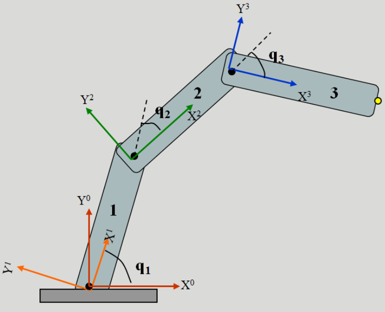


Figure: robot task

Homogeneous transformations

Consider the planar PRP robot with $n = 3$ joints in the Figure below. The world reference frame $RF_w = (x_w, y_w, z_w)$ and the end-effector frame $RF_e = (x_e, y_e, z_e)$ are also shown.

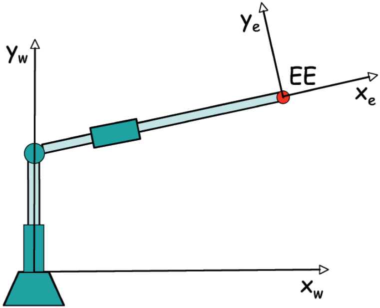
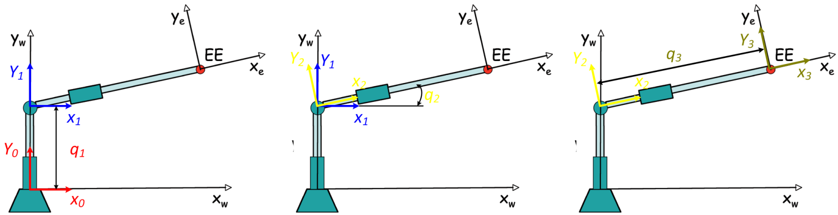


Figure: robot task

Homogeneous transformations

$$
{}^0\mathbf{T}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_2 & \sin q_2 & 0 & 0 \\ -\sin q_2 & \cos q_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & q_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0\mathbf{T}^3 = {}^0\mathbf{T}^1 \; {}^1\mathbf{T}^2 \; {}^2\mathbf{T}^3 = \begin{bmatrix} \cos q_2 & \sin q_2 & 0 & q_3 \cos q_2 \\ -\sin q_2 & \cos q_2 & 0 & q_1 - q_3 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



$$^{0}\mathbf{T}^{3} = {}^{0}\mathbf{T}^{1}\,{}^{1}\mathbf{T}^{2}\,{}^{2}\mathbf{T}^{3} = \begin{bmatrix} \cos q_2 & \sin q_2 & 0 & q_3 \cos q_2 \\ -\sin q_2 & \cos q_2 & 0 & q_1 - q_3 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For the task involving only the end-effector position on the plane, it is

$$\varphi(\mathbf{q}) = \begin{bmatrix} {}^{w}p_x \\ {}^{w}p_y \end{bmatrix} = \begin{bmatrix} q_3 \cos q_2 \\ q_1 - q_3 \sin q_2 \end{bmatrix}$$

Jacobian matrix is

$$\mathbf{J}(\mathbf{q}) = \frac{\partial \varphi(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} 0 & -q_3 \sin q_2 & \cos q_2 \\ 1 & -q_3 \cos q_2 & -\sin q_2 \end{bmatrix}$$

this matrix looses rank if and only if $q_3 = 0$ and $\cos q_2 = 0$, i.e., when the third robot link is oriented along the vertical direction and the third joint is completely retracted. In such a configuration, the Jacobian becomes

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & \pm 1 \end{bmatrix}$$

Homogeneous transformations

For the end-effector planar positioning and orientation task (of dimension $m = 3$), it is

$$\varphi(\mathbf{q}) = \begin{bmatrix} {}^{w}p_x \\ {}^{w}p_y \\ \phi \end{bmatrix} = \begin{bmatrix} q_3 \cos q_2 \\ q_1 - q_3 \sin q_2 \\ q_2 \end{bmatrix}$$

Jacobian matrix is

$$\mathbf{J}(\mathbf{q}) = \frac{\partial \varphi(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} 0 & -q_3 \sin q_2 & \cos q_2 \\ 1 & -q_3 \cos q_2 & -\sin q_2 \\ 0 & 1 & 0 \end{bmatrix}$$

is singular if and only if $\cos q_2 = 0$

Feedback Stabilization

Let us express the kinematic equation in the following form: For the task involving only the end-effector position on the plane, it is

$$\mathbf{x} = \varphi(\mathbf{q})$$

In solving the inverse kinematics problem, we assume that an error $\mathbf{e}$ exist. Therefore,

$$\mathbf{x} - \varphi(\mathbf{q}) = \mathbf{e}$$

Now, let us select a positive-definite Lyapunov function $V(\mathbf{e})$ as $V(\mathbf{e}) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e}$. The time-derivative of this Lyapunov function is given by

$$\dot{V}(\mathbf{e}) = \frac{1}{2}\dot{\mathbf{e}}^{\mathrm{T}}\mathbf{e} + \frac{1}{2}\mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}} = \mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}}$$

We know that $\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\varphi}(\mathbf{q}) = -\dot{\varphi}(\mathbf{q}) = -\frac{\partial\varphi(\mathbf{q})}{\partial\mathbf{q}}\frac{\partial\mathbf{q}}{\partial t} = -\mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$, (we assume that the end-effector moves slowly). Therefore, the time-derivative of $V(\dot{\mathbf{e}})$ is

$$\dot{V}(\mathbf{e}) = \mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}} = -\mathbf{e}^{\mathrm{T}}\mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

$$\dot{V}(\mathbf{e}) = \mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}} = -\mathbf{e}^{\mathrm{T}}\mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

Let us set $\dot{\mathbf{q}} \rightarrow \mathbf{J}(\mathbf{q})^{-1}\mathbf{ke}$ This yields

$$\dot{V}(\mathbf{e}) = -\mathbf{e}^{\mathrm{T}}\mathbf{J}(\mathbf{q})\mathbf{J}(\mathbf{q})^{-1}\mathbf{ke} = -\mathbf{e}^{\mathrm{T}}\mathbf{ke}$$

Therefore, **k** must be positive-definite. Now, the error is stable in the sense of Lyapunov.

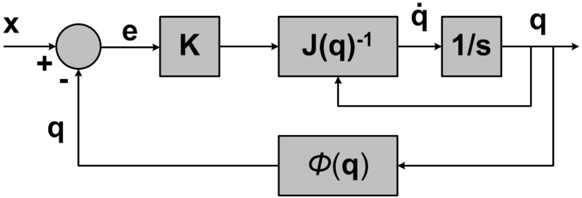


Figure: Feedback Stabilization

Thanks

Questions please