Robust and Optimal Control

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Necessary conditions for optimal control

The problem is to find an optimal control (u^*) that causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \qquad (1)$$

to follow a trajectory (x^*) that minimizes the performance measure

$$J(\mathbf{x}(t), t) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau. \qquad (2)$$

Please refer to chapter 5, page 184 (Kirk).

Necessary conditions for optimal control

The Hamiltonian is given by

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^{\mathrm{T}}(t)\left[\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)\right]. \tag{3}$$

The necessary conditions of optimality are

$$\dot{\mathbf{x}}^*(t) = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t), \tag{4}$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t), \tag{5}$$

$$0 = \frac{\partial \mathcal{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t). \tag{6}$$

Example

The system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) && (7) \\ \dot{x}_2(t) &= -x_2(t) + u(t) && (8) \end{aligned}$$

is to be controlled so that its control effort is conserved. Therefore, the performance measure is given by

$$J(u) = \int_{t_0}^{t_f} \frac{1}{2} u^2(t) dt. \quad (9)$$

The Hamiltonian (36) is given by

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = \frac{1}{2} u^2(t) + p_1(t)x_2(t) - p_2(t)x_2(t) + p_2(t)u(t). \quad (10)$$

Example

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = \frac{1}{2}u^2(t) + p_1(t)x_2(t) - p_2(t)x_2(t) + p_2(t)u(t). \tag{11}$$

The necessary conditions for optimality are

$$\dot{p}_1^*(t) = \frac{\partial \mathcal{H}}{\partial x_1} = 0 \tag{12}$$

$$\dot{p}_2^*(t) = -\frac{\partial \mathcal{H}}{\partial x_2} = -p_1^*(t) + p_2^*(t), \tag{13}$$

and

$$0 = \frac{\partial \mathcal{H}}{\partial u} = u^*(t) + p_2^*(t). \tag{14}$$

The plant is described by the linear state equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{15}$$

which may have time-varying coefficients. The performance measure to be minimized is

$$J = \frac{1}{2}\mathbf{x}^{\mathrm{T}}(t_f)\mathbf{H}\mathbf{x}(t_f) + \int_{t_0}^{t_f} \frac{1}{2}\left[\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u}\right] dt, \tag{16}$$

where the final time t_f is fixed. Further, $\mathbf{H}$ and $\mathbf{Q}$ are real symmetric positive semi-definite matrices. Finally, $\mathbf{R}$ is a real symmetric positive definite matrix.
The Hamiltonian is

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^{\mathrm{T}}(t)\left[\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)\right]. \tag{17}$$

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u} + \mathbf{p}^{\mathrm{T}}\mathbf{A}(t)\mathbf{x}(t) + \mathbf{p}^{\mathrm{T}}\mathbf{B}(t)\mathbf{u}(t), \tag{18}$$

and the necessary conditions for optimality are

$$\dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) + \mathbf{B}(t)\mathbf{u}^*(t), \tag{19}$$

$$\dot{\mathbf{p}}^*(t) = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}^{\mathrm{T}}(t)\mathbf{p}^*(t), \tag{20}$$

$$0 = \mathbf{R}(t)\mathbf{u}^*(t) + \mathbf{B}^{\mathrm{T}}(t)\mathbf{p}^*(t). \tag{21}$$

Solving (54) for $\mathbf{u}^*(t)$ yields

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{p}^*(t). \tag{22}$$

Substitution of (55) into (52) yields

$$\dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{p}^*(t). \tag{23}$$

Putting (53) and (56) into the following matrix format

$$\begin{bmatrix} \dot{\mathbf{x}}^*(t) \\ -- \\ \dot{\mathbf{p}}^*(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(t) & | & -\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t) \\ -- & -- & -- \\ -\mathbf{Q}(t) & | & \mathbf{p}^*(t) \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(t) \\ -- \\ \mathbf{p}^*(t) \end{bmatrix}. \quad (24)$$

The solution of these equations has the following form

$$\begin{bmatrix} \mathbf{x}^*(t_f) \\ -- \\ \mathbf{p}^*(t_f) \end{bmatrix} = \varphi(t_f, t) \begin{bmatrix} \mathbf{x}^*(t) \\ -- \\ \mathbf{p}^*(t) \end{bmatrix}, \quad (25)$$

where $\varphi(t_f, t)$ is the state-transition matrix of the system (57).

$$\begin{bmatrix} \mathbf{x}^*(t_f) \\ -- \\ \mathbf{p}^*(t_f) \end{bmatrix} = \begin{bmatrix} \varphi_{11}(t_f, t) & | & \varphi_{12}(t_f, t) \\ -- & -- & -- \\ \varphi_{21}(t_f, t) & | & \varphi_{22}(t_f, t) \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(t) \\ -- \\ \mathbf{p}^*(t) \end{bmatrix}, \quad (26)$$

From the boundary condition, the final co-states are related to the final states using

$$\mathbf{p}^*(t_f) = \mathbf{H}\mathbf{x}^*(t_f). \tag{27}$$

Solving for $\mathbf{p}^*(t_f)$, we obtain

$$\mathbf{p}^*(t) = \left[\varphi_{22}(t_f, t) - \mathbf{H}\varphi_{12}(t_f, t)\right]^{-1}\left[\mathbf{H}\varphi_{11}(t_f, t) - \varphi_{21}(t_f, t)\right]\mathbf{x}^*(t). \tag{28}$$

$$\mathbf{p}^*(t) = \mathbf{K}(t)\mathbf{x}^*(t). \tag{29}$$

The optimal control law is given by

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{K}(t)\mathbf{x}^*(t). \tag{30}$$

Example

It is desired to determine the control law (using the principle of optimality and the Hamilton-Jacobi-Bellman equation) that causes the plant

$$\dot{x}_1 = x_2(t) \quad (31)$$

$$\dot{x}_2 = -x_1(t) - 2x_2(t) + u(t) \quad (32)$$

to minimize the performance measure

$$J = 10x_1^2(T) + \frac{1}{2}\int_0^T \left[x_1^2(t) + 2x_2^2(t) + u^2(t)\right]. \quad (33)$$

Consider the scalar case

$$\dot{x}(t) = ax(t). \tag{34}$$

Taking the Laplace transform of (34), we obtain

$$sX(s) - x(0) = aX(s), \tag{35}$$

$$X(s) = \frac{x(0)}{s-a} = (s-a)^{-1}x(0). \tag{36}$$

Finally, inverse Laplace transform of (36) yields

$$x(t) = e^{at}x(0). \tag{37}$$

Now consider the following homogenous state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t). \tag{38}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s), \tag{39}$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0). \tag{40}$$

The inverse Laplace transform yields

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]\mathbf{x}(0) = e^{\mathbf{A}t}\mathbf{x}(0). \tag{41}$$

Therefore, the state transition matrix ($e^{\mathbf{A}t}$) is given by

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]. \tag{42}$$

- Calculate the state transition matrix of the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (43)$$

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} (s+1) & 0 \\ -2 & (s+3) \end{bmatrix} \quad (44)$$

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \frac{(s+3)}{(s+1)(s+3)} & 0 \\ \frac{2}{(s+1)(s+3)} & \frac{(s+1)}{(s+1)(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(s+1)} & 0 \\ \left(\frac{1}{(s+1)} - \frac{1}{(s+1)}\right) & \frac{1}{(s+3)} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right], \quad (45)$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & 0 \\ (e^{-t} - e^{-3t}) & e^{-3t} \end{bmatrix}.$$

- Calculate the state transition matrix of the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (46)$$

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & -1 \\ 2 & (s+3) \end{bmatrix} \qquad (47)$$

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \frac{(s+3)}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right], \qquad (48)$$

$$= \begin{bmatrix} 2e^{t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}.$$

State Transition Matrix

If the matrix $\mathbf{A}$ can be transformed into a diagonal form, then the state transition matrix $e^{\mathbf{A}t}$ is given by

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \mathbf{P}\begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \dots & \ddots & 0 \\ 0 & \dots & 0 & e^{\lambda_n t} \end{bmatrix}\mathbf{P}^{-1}, \qquad (49)$$

where $\mathbf{P}$ is a digonalizing matrix for $\mathbf{A}$. Further, λ_i is the ith eigenvalue of the matrix $\mathbf{A}$, for $i = 1, \dots, n$.

Derivation: Consider the following homogenous state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{50}$$

and the following similarity transformation:

$$\mathbf{x} = \mathbf{P}\xi \quad , \quad \dot{\mathbf{x}} = \mathbf{P}\dot{\xi}. \tag{51}$$

Substituting (51) in (50) yields

$$\dot{\xi} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\xi = \mathbf{D}\xi. \tag{52}$$

Solution of (52) is

$$\xi(t) = e^{\mathbf{D}t}\xi(0), \tag{53}$$

using (51)

$$\mathbf{x}(t) = \mathbf{P}\xi(t) = \mathbf{P}e^{\mathbf{D}t}\xi(0) \quad , \quad \mathbf{x}(0) = \mathbf{P}\xi(0). \tag{54}$$

Therefore

$$\mathbf{x}(t) = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\mathbf{x}(0) = e^{\mathbf{A}t}\mathbf{x}(0). \tag{55}$$

- Calculate the state transition matrix of the following system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{56}$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 0$ and $\lambda_2 = -2$. A similarity transformation matrix $\mathbf{P}$ is

$$\mathbf{P} = \begin{bmatrix}1 & 1\\ 0 & -2\end{bmatrix}. \tag{57}$$

Using (49) to calculate the state transition matrix

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} \tag{58}$$

$$= \begin{bmatrix}1 & 1\\ 0 & -2\end{bmatrix}\begin{bmatrix}e^0 & 0\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}1 & \frac{1}{2}\\ 0 & -\frac{1}{2}\end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix}1 & \frac{1}{2}(1 - e^{-2t})\\ 0 & e^{-2t}\end{bmatrix}. \tag{59}$$

Consider the state equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad (60)$$

and the performance measure to be minimized is

$$J = \frac{1}{2}\left[\mathbf{x}(t_f) - \mathbf{r}(t_f)\right]^{\mathrm{T}} \mathbf{H} \left[\mathbf{x}(t_f) - \mathbf{r}(t_f)\right] + \int_{t_0}^{t_f} \frac{1}{2}\left[(\mathbf{x}-\mathbf{r})^{\mathrm{T}}\mathbf{Q}(\mathbf{x}-\mathbf{r}) + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u}\right] dt, \quad (61)$$

where $r(t)$ is the desired reference value and the final time t_f is fixed. Further, **H** and **Q** are real symmetric positive semi-definite matrices. Finally, **R** is a real symmetric positive definite matrix. The Hamiltonian is given by

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = \frac{1}{2}\left[(\mathbf{x}-\mathbf{r})^{\mathrm{T}}\mathbf{Q}(\mathbf{x}-\mathbf{r}) + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u}\right] + \mathbf{p}^{\mathrm{T}}(t)\mathbf{A}(t)\mathbf{x}(t) + \mathbf{p}^{\mathrm{T}}(t)\mathbf{B}(t)\mathbf{u}(t). \quad (62)$$

The necessary conditions of optimality are

$$\begin{aligned}\dot{\mathbf{x}}^*(t) &= \frac{\partial \mathcal{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t), \\ &= \mathbf{A}(t)\mathbf{x}^*(t) + \mathbf{B}(t)\mathbf{u}^*(t).\end{aligned} \tag{63}$$

$$\begin{aligned}\dot{\mathbf{p}}^*(t) &= -\frac{\partial \mathcal{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t), \\ &= -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}^{\mathrm{T}}(t)\mathbf{p}^*(t) + \mathbf{Q}(t)\mathbf{r}(t).\end{aligned} \tag{64}$$

$$\begin{aligned}0 &= \frac{\partial \mathcal{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t), \\ &= \mathbf{R}(t)\mathbf{u}^*(t) + \mathbf{B}^{\mathrm{T}}(t)\mathbf{p}^*(t).\end{aligned} \tag{65}$$

Therefore

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{p}^*(t). \tag{66}$$

Substituting (66) in (63) yields

$$\begin{bmatrix} \dot{\mathbf{x}}^*(t) \\ -- \\ \dot{\mathbf{p}}^*(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(t) & | & -\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t) \\ -- & - & -- \\ -\mathbf{Q}(t) & | & \mathbf{p}^*(t) \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(t) \\ -- \\ \mathbf{p}^*(t) \end{bmatrix} + \begin{bmatrix} 0 \\ - \\ \mathbf{Q}(t)\mathbf{r}(t) \end{bmatrix}. \quad (67)$$

Solution of (67) is

$$\begin{bmatrix} \mathbf{x}^*(t_f) \\ -- \\ \mathbf{p}^*(t_f) \end{bmatrix} = \varphi(t_f, t) \begin{bmatrix} \mathbf{x}^*(t) \\ -- \\ \mathbf{p}^*(t) \end{bmatrix} + \int_t^{t_f} \varphi(t_f, \tau) \begin{bmatrix} 0 \\ - \\ \mathbf{Q}(\tau)\mathbf{r}(\tau) \end{bmatrix} d\tau, \quad (68)$$

where φ is the state transition matrix. If φ is partitioned, we can replace the integral with a $2n$ vector

$$\begin{bmatrix} \mathbf{x}^*(t_f) \\ -- \\ \mathbf{p}^*(t_f) \end{bmatrix} = \varphi(t_f, t) \begin{bmatrix} \mathbf{x}^*(t) \\ -- \\ \mathbf{p}^*(t) \end{bmatrix} + \begin{bmatrix} \mathbf{f}_1(t) \\ - \\ \mathbf{f}_2(t) \end{bmatrix}. \quad (69)$$

$$\begin{bmatrix} \mathbf{x}^*(t_f) \\ -- \\ \mathbf{p}^*(t_f) \end{bmatrix} = \begin{bmatrix} \varphi_{11}(t_f, t) & | & \varphi_{12}(t_f, t) \\ -- & - & -- \\ \varphi_{21}(t_f, t) & | & \varphi_{22}(t_f, t) \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(t) \\ -- \\ \mathbf{p}^*(t) \end{bmatrix} + \begin{bmatrix} \mathbf{f}_1(t) \\ - \\ \mathbf{f}_2(t) \end{bmatrix}. \tag{70}$$

The boundary conditions are

$$\mathbf{p}^*(t_f) = \mathbf{H}\mathbf{x}^*(t_f) - \mathbf{H}\mathbf{r}(t_f). \tag{71}$$

Solving for $\mathbf{p}^*(t_f)$, we obtain

$$\begin{aligned} \mathbf{p}^*(t) \quad = \quad & [\varphi_{22}(t_f, t) - \mathbf{H}\varphi_{12}(t_f, t)]^{-1} [\mathbf{H}\varphi_{11}(t_f, t) - \varphi_{21}(t_f, t)] \, \mathbf{x}^*(t) + \\ & [\varphi_{22}(t_f, t) - \mathbf{H}\varphi_{12}(t_f, t)]^{-1} [\mathbf{H}\mathbf{f}_1(t) - \mathbf{H}\mathbf{r}(t_f) - \mathbf{f}_2(t)] \end{aligned} \tag{72}$$

$$\mathbf{p}^*(t) = \mathbf{K}(t)\mathbf{x}^*(t) + \mathbf{s}(t). \tag{73}$$

The optimal control law is given by

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{K}(t)\mathbf{x}(t) - \mathbf{R}^{-1}(t)\mathbf{B}^{-1}(t)\mathbf{s}(t). \tag{74}$$

Thanks

Questions please