Assignment 3 - Deadline 1 week

Problem 1
For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable:

1. $\dot{x}_1 = -x_1 + x_1 x_2 \quad , \quad \dot{x}_2 = -x_2$
2. $\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2) \quad , \quad \dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$
3. $\dot{x}_1 = x_2(1 - x_1^2) \quad , \quad \dot{x}_2 = -(x_1 + x_2)(1 - x_1^2)$
4. $\dot{x}_1 = -x_1 - x_2 \quad , \quad \dot{x}_2 = 2x_1 - x_2^3$

Investigate if the origin is globally asymptotically stable.

Problem 2
For each of the following systems, show that the origin is unstable:

1. $\dot{x}_1 = x_1^3 + x_1^2 x_2 \quad , \quad \dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$
2. $\dot{x}_1 = -x_1^3 + x_2 \quad , \quad \dot{x}_2 = x_1^6 - x_2^3$

Problem 3
Show that the origin of

$$\dot{x}_1 = x_2 \quad , \quad \dot{x}_2 = -x_1^3 - x_2^3$$

is globally asymptotically stable.

Problem 4
Consider the system defined by the following equations:

$$\dot{x}_1 = x_2 + \beta\left(\frac{x_1^3}{3} - x_1\right), \quad (1)$$

$$\dot{x}_2 = -x_1. \quad (2)$$

Describe the stability of the equilibrium point (origin) using Lyapunov theory, in the following cases:

- $\beta > 0$;
- $\beta = 0$;
- $\beta < 0$.

Problem 5

Consider the system defined by the following equations:

$$\dot{x}_1 = x_2 + a x_1 \sin x_1, \quad (3)$$
$$\dot{x}_2 = b x_1 x_2 + u. \quad (4)$$

where the variables a and b are unknown constants that are bounded using

$$| a - 1 | \leq 1 \text{ and } | b - 1 | \leq 2. \quad (5)$$

Using Lyapunov theory, design a continuous globally stabilizing state feedback control system using the control input $u(t)$.

Problem 6

Given the following system:

$$\dot{x}_1 = x_1^3 + x_2, \quad (6)$$
$$\dot{x}_2 = x_1^2 x_2 - x_1^4 + u. \quad (7)$$

Using the Lyapunov theory outlined in class, design a state feedback control law to stabilize the equilibrium point at the origin. Discuss stability and show that all signals are bounded.

Problem 7

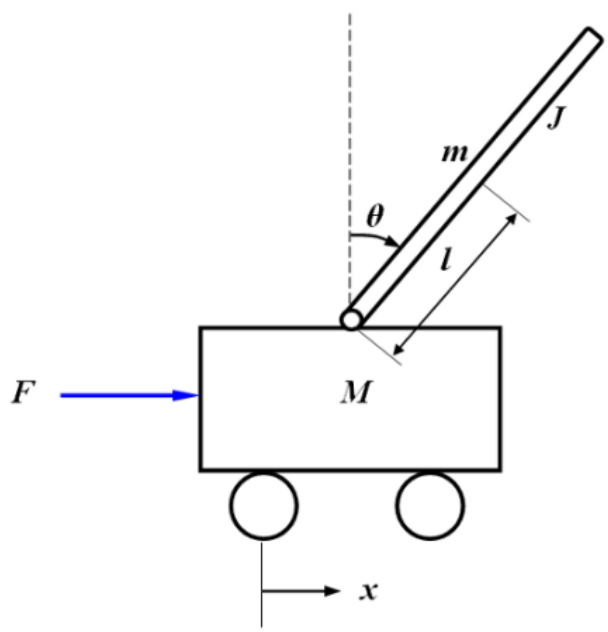

Figure 1: The cart-pendulum system.

Figure 1 shows the physical model of a cart-pendulum system. M and m are the masses of the cart and the inverted pendulum, respectively. l is the distance from the center of gravity of the link to its attachment point. The coordinate x represents the position of the cart on the horizontal axis to a fixed point, and θ is the rotational angle of the pendulum. Using the Lagrangian equations, one can show that the dynamic equations of the cart-pendulum are given by

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F, \tag{8}$$

$$\frac{4}{3}ml\ddot{\theta} + m\ddot{x}\cos\theta - mg\sin\theta = 0, \tag{9}$$

Design an observer-based control system to stabilize the cart-pendulum in a position where the pendulum is in the unstable vertical position ($\theta = 0$) and the cart is at a given point on the straight line, $x = 0$, under the control action of the control force F. It is also required to achieve the following:

- Design a non-linear state observer of the system and provide a stability proof for the estimation error dynamics;
- Simulate the estimation of the states using Simulink;
- Add a deviation of 5$\%$ to the nominal M and m used in observer and simulate the dynamics of the system;
- Change the initial conditions of the observer and simulate the results of the state-observer;
- Prove that the devised observer-based control system achieves stability based on Lyapunov theory;
- Simulate the dynamics of the controlled cart-pendulum and show the states, errors, and control force;
- Design a control system that stabilizes the origin (or equilibrium point) of the system.