Consider a nonlinear MIMO system of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) = f(x) + G(x)u \tag{1a}$$

$$y_1 = h_1(x) \tag{1b}$$

$$\vdots \tag{1c}$$

$$y_m = h_m(x) \tag{1d}$$

For simplicity, we consider systems with the same number of inputs and outputs. For convenience, we might use $y = (y_1, \ldots, y_m)$ and $h = (h_1, \ldots, h_m)$.

We are interested in finding an appropriate change of coordinates and a static state feedback that renders the system linear. Let us begin by generalizing the notion of relative degree of SISO systems.

Vector relative degree

Definition 2.1 (Vector relative degree) *A MIMO system* (1) *has vector relative degree* $(\rho_1, \ldots, \rho_m)$ *in a region* $D_0 \subset D$ *if, for all* $x \in D_0$,

$$\mathcal{L}_{g_j}\mathcal{L}_f^{k-1}h_i(x) = 0, \quad j = 1, \ldots, m,\ k = 1, \ldots, \rho_i - 1,\ i = 1, \ldots, m.$$

and the matrix

$$A(x) = \begin{pmatrix} \mathcal{L}_{g_1}\mathcal{L}_f^{\rho_1-1}h_1(x) & \ldots & \mathcal{L}_{g_m}\mathcal{L}_f^{\rho_1-1}h_1(x) \\ \mathcal{L}_{g_1}\mathcal{L}_f^{\rho_2-1}h_2(x) & \ldots & \mathcal{L}_{g_m}\mathcal{L}_f^{\rho_2-1}h_2(x) \\ \ldots & \ldots & \ldots \\ \mathcal{L}_{g_1}\mathcal{L}_f^{\rho_m-1}h_m(x) & \ldots & \mathcal{L}_{g_m}\mathcal{L}_f^{\rho_m-1}h_m(x) \end{pmatrix} \tag{2}$$

is nonsingular.

Example **(Rigid body rotation [2])** Consider the following model of a rigid body whose gas jets control the rotations around the two first principal axes

$$\begin{aligned}\dot{\omega}_1 &= a_1\omega_2\omega_3 + u_1\\ \dot{\omega}_2 &= a_2\omega_1\omega_3 + u_2\\ \dot{\omega}_3 &= a_3\omega_1\omega_2\end{aligned}$$

Alternatively, we have $x = (\omega_1, \omega_2, \omega_3)$, $f = (a_1\omega_2\omega_3, a_2\omega_1\omega_3, a_3\omega_1\omega_2)$, $g_1 = (1, 0, 0)$, and $g_2 = (0, 1, 0)$. Let us consider as outputs $y_1 = \omega_1$ and $y_2 = \omega_2$. It is not difficult to see that the system has vector relative degree $(1, 1)$. •

Exercise Using the Definition 2.1, show that, if (1) has relative degree $(\rho_1, \ldots, \rho_m)$, then, for each $i = 1, \ldots, m$, there exists at least one $j \in \{1, \ldots, m\}$ such that the SISO system $\dot{x} = f + g_j u_j$, $y_i = h_i(x)$ has relative degree ρ_i. Moreover, for any other j, the corresponding relative degree - if it exists - is necessarily higher than or equal to ρ_i. •

The next result shows that the relative degree helps us find the right change of coordinates to turn the system into linear form.

Theorem *Assume the MIMO system* (1) *has a relative degree* $(\rho_1, \ldots, \rho_m)$ *in* D_0. *Then* $\rho = \rho_1 + \cdots + \rho_m \leq n$. *Set, for each* $i \in \{1, \ldots, m\}$,

$$\begin{aligned} \phi_1^i &= h_i \\ \phi_2^i &= \mathcal{L}_f h_i \\ &\vdots \\ \phi_{\rho_i}^i &= \mathcal{L}_f^{\rho_i - 1} h_i \end{aligned}$$

If $\rho = n$, *then this defines a local diffeomorphism. If* $\rho < n$, *then one can find* $n - \rho$ *functions* $\phi_{\rho+1}, \ldots, \phi_n$ *such that*

$$T(x) = (\phi_1^1, \ldots, \phi_{\rho_1}^1, \ldots, \phi_1^m, \ldots, \phi_{\rho_m}^m, \phi_{\rho+1}, \ldots, \phi_n)$$

is a local diffeomorphism. Moreover, if the input distribution $\Delta = \mathrm{span}\{g_1, \ldots, g_m\}$ *is involutive, it is always possible to find the* $n - \rho$ *functions* $\phi_{\rho+1}, \ldots, \phi_n$ *such that*

$$\mathcal{L}_{g_j} \phi_i = 0, \quad j = 1, \ldots, m, \; i = \rho + 1, \ldots, m. \tag{3}$$

Exercise In the SISO case, why is it always possible to choose the $n-\rho$ functions such that (3) is satisfied? •

Let us now compute the form of the equations in the new coordinates given by Theorem . For $i \in \{1,\dots,m\}$, let

$$\xi^i = \begin{pmatrix} \xi_1^i \\ \xi_2^i \\ \vdots \\ \xi_{\rho_i}^i \end{pmatrix} = \begin{pmatrix} \phi_1^i \\ \phi_2^i \\ \vdots \\ \phi_{\rho_i}^i \end{pmatrix}$$

and define

$$\xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^m \end{pmatrix}, \quad \eta = \begin{pmatrix} \phi_{\rho+1} \\ \vdots \\ \rho_n \end{pmatrix}$$

In the new coordinates (ξ,η), the original control system takes the form

$$\dot{\xi}_1^i = \xi_2^i \tag{4a}$$

$$\dot{\xi}_2^i = \xi_3^i \tag{4b}$$

$$\vdots \tag{4c}$$

$$\dot{\xi}_{\rho_i-1}^i = \xi_{\rho_i}^i \tag{4d}$$

$$\dot{\xi}_{\rho_i}^i = \mathcal{L}_f^{\rho_i} h_i + \sum_{j=1}^{m} \mathcal{L}_{g_j}\mathcal{L}_f^{\rho_i-1} h_i\, u_j, \qquad 1 \le i \le m, \tag{4e}$$

$$\dot{\eta} = q(\xi,\eta) + \sum_{j=1}^{m} p_j(\xi,\eta) u_j, \tag{4f}$$

$$y_i = \xi_1^i, \qquad 1 \le i \le m. \tag{4g}$$

Note that in general the inputs appear explicitly in the equation for η. However, if the input distribution is involutive, then the result states that we can always choose the functions $\phi_{\rho+1},\dots,\phi_n$ such that the equation for η reads $\dot{\eta} = q(\xi,\eta)$.

Zero dynamics

Given our discussion above, it is not difficult now to obtain an expression for the zero dynamics in the new coordinates (ξ, η). We set

$$\xi = 0.$$

Looking at (4), $\xi^i_{\rho_i}$, $i \in \{1, \dots, m\}$ remains at zero only if

$$0 = \mathcal{L}_f^{\rho_i} h_i + \sum_{j=1}^{m} \mathcal{L}_{g_j} \mathcal{L}_f^{\rho_i - 1} h_i \, u_j, \qquad 1 \leq i \leq m.$$

Because of the definition of vector relative degree, we know that this equation has a unique solution for $u = (u_1, \dots, u_m)$. More specifically, we have

$$u = -A(x)^{-1} b(x),$$

where

$$b(x) = \begin{pmatrix} \mathcal{L}_f^{\rho_1} h_1 \\ \vdots \\ \mathcal{L}_f^{\rho_m} h_m \end{pmatrix}$$

The zero dynamics therefore looks like

$$\dot{\eta} = q(0, \eta) - p(0, \eta) A(0, \eta)^{-1} b(0, \eta).$$

Full-state exact linearization via feedback

The full-state exact linearization problem we are interested in solving can be formulated as follows: given (1), find a static state feedback

$$u(x) = \alpha(x) + \beta(x)v$$

where $\alpha(x) \in \mathbb{R}^m$ and $\beta(x) \in \mathbb{R}^{m\times m}$ is invertible and a change of coordinates $z = T(x)$ such that, in the new coordinates and with the feedback u, the system reads

$$\begin{aligned} \dot{z} &= Az + Bu \\ y_1 &= \tilde{h}_1(z) \\ &\vdots \\ y_m &= \tilde{h}_m(z) \end{aligned}$$

By looking at Theorem 2.4, one can deduce that the problem is solvable if the system (1) has vector relative degree $(\rho_1, \ldots, \rho_m)$ such that $\rho = \rho_1 + \cdots + \rho_m = n$. For, assuming this is the case, then selecting the change of coordinates given by the theorem puts the system into the form

$$\begin{aligned} \dot{\xi}^i_1 &= \xi^i_2 \\ \dot{\xi}^i_2 &= \xi^i_3 \\ &\vdots \\ \dot{\xi}^i_{\rho_i-1} &= \xi^i_{\rho_i} \\ \dot{\xi}^i_{\rho_i} &= \mathcal{L}_f^{\rho_i} h_i + \sum_{j=1}^{m} \mathcal{L}_{g_j}\mathcal{L}_f^{\rho_i-1} h_i\, u_j \\ y_i &= \xi^i_1 \end{aligned}$$

for $i \in \{1, \ldots, m\}$. Selecting now the feedback controller

$$u = A(x)^{-1}(-b(x) + v)$$

yields

$$\begin{aligned}
\dot{\xi}_1^i &= \xi_2^i \\
\dot{\xi}_2^i &= \xi_3^i \\
&\vdots \\
\dot{\xi}_{\rho_i-1}^i &= \xi_{\rho_i}^i \\
\dot{\xi}_{\rho_i}^i &= v_i \\
y_i &= \xi_1^i
\end{aligned}$$

The following result that our assumptions are not only sufficient but indeed necessary.

Proposition *The full-state exact linearization problem is solvable on a domain D_0 if and only if there exist output functions $h_1, \ldots, h_m : \mathbb{R}^n \to \mathbb{R}$ such that the system (1) has vector relative degree $(\rho_1, \ldots, \rho_m)$ on D_0 with $\rho_1 + \cdots + \rho_m = n$.*

Interestingly, the existence of such desirable outputs can be fully characterized in terms of the vector fields $f, g_1, \ldots, g_m$ defining the MIMO system. The following is the MIMO version of the SISO result of Lecture 2.

Theorem *Consider the MIMO system on D*

$$\dot{x} = f(x) + \sum_{i=1}^{p} u_i g_i(x),$$

and assume the control distribution has maximal rank. Define the following distributions

$$\begin{aligned}
\Delta_0 &= \operatorname{span}\{g_1, \ldots, g_m\} \\
\Delta_1 &= \operatorname{span}\{g_1, \ldots, g_m, \operatorname{ad}_f g_1, \ldots, \operatorname{ad}_f g_m\} \\
&\vdots \\
\Delta_i &= \operatorname{span}\{\operatorname{ad}_f^k g_j \mid 0 \le k \le i,\ 1 \le j \le m\}.
\end{aligned}$$

Then the full-state exact linearization problem is solvable if and only if the following conditions hold

(i) Δ_i, $0 \le i \le n-1$, *has constant dimension;*

(ii) Δ_i, $0 \le i \le n-2$, *is involutive;*

(iii) Δ_{n-1} *has dimension* n.

Noninteracting control problem

We may wish to use feedback also to "reduce the system" to an aggregate of independent single-input single-output channels. This is the problem known as *noninteracting control*. Surprisingly, we will realize that the notion of vector relative degree is powerful enough to deal with the problem in an elegant way. Roughly speaking, given a MIMO system (1), we look for a feedback controller $u = \alpha(x) + \beta(x)v$ such that, in the resulting system, the output y_i is affected only by the corresponding input v_i, and not by v_j, for $j \neq i$.

Let us start with sufficient conditions to solve the noninteracting control problem. Assume the MIMO system (1) has relative degree $(\rho_1, \ldots, \rho_m)$ (it is not necessary that $\rho_1 + \cdots + \rho_m = n$). Write the system in its normal form (4) in the coordinates (ξ, η). Choosing the controller $u = A(x)^{-1}(-b(x) + v)$ yields

$$
\begin{aligned}
\dot{\xi}_1^i &= \xi_2^i \\
\dot{\xi}_2^i &= \xi_3^i \\
&\ \vdots \\
\dot{\xi}_{\rho_i-1}^i &= \xi_{\rho_i}^i \\
\dot{\xi}_{\rho_i}^i &= v_i \\
y_i &= \xi_1^i
\end{aligned}
$$

for $1 \leq i \leq m$ together with

$$
\dot{\eta} = q(\xi, \eta) - p(\xi, \eta)A(\xi, \eta)^{-1}b(\xi, \eta) + p(\xi, \eta)A(\xi, \eta)^{-1}v.
$$

The structure of these equations show that the noninteracting requirement has been achieved!

The normal form is helpful in order to see how this works. However, the achievement of this input-output noninteractive behavior is independent of the coordinates chosen (do you see why?).

Indeed, these sufficient conditions are also necessary, as the following result states.

Proposition *Consider the MIMO system (1). The noninteractive control problem is solvable if and only if the system has some vector relative degree.*

What if no relative degree exists?

The previous discussion has assumed that a vector relative degree exists. What if this is not the case? Let us start by considering an illustrative example.

An illustrative example: a car-like robot

Following [3], consider a nonholonomic car-like robot with rear-wheel driving as depicted in Figure 1. The dynamics of the system is

$$\begin{aligned}\dot{x} &= v_1 \cos\theta \\ \dot{y} &= v_1 \sin\theta \\ \dot{\theta} &= v_1 (\tan\phi)/\ell \\ \dot{\phi} &= v_2\end{aligned}$$

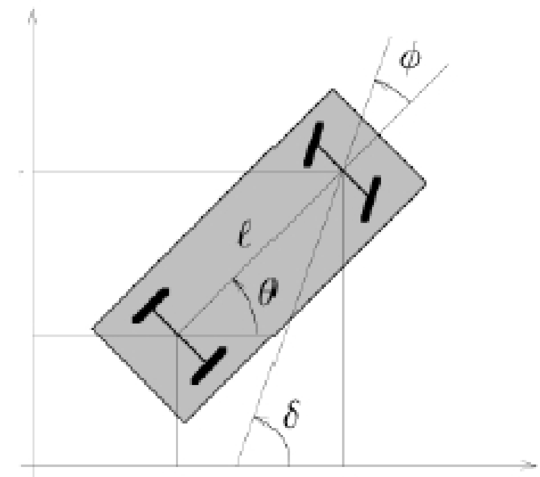

Figure 1: Nonholonomic car-like robot

Here (x, y) are the coordinates of the center of mass of the rear wheel axis, v_1 is the driving velocity input and v_2 is the steering velocity input. Note that, with $\bar{x} = (x, y, \theta, \phi)$, $g_1 = (\cos\theta, \sin\theta, \tan\phi/\ell, 0)$ and $g_2 = (0, 0, 0, 1)$, the system can be written as

$$\dot{\bar{x}} = v_1 g_1(\bar{x}) + v_2 g_2(\bar{x}).$$

For trajectory tracking problems, the natural outputs to consider are the position coordinates

$$h_1(x, y, \theta, \phi) = x, \qquad h_2(x, y, \theta, \phi) = y.$$

Let us compute the relative degree of the system. Note that $f = 0$. Therefore, we have

$$\begin{aligned} \mathcal{L}_{g_1} h_1 &= \cos\theta, \quad \mathcal{L}_{g_2} h_1 = 0, \\ \mathcal{L}_{g_1} h_2 &= \sin\theta, \quad \mathcal{L}_{g_2} h_2 = 0. \end{aligned}$$

Since the matrix

$$A(x) = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{pmatrix}$$

is singular, we conclude that the system does not have a vector relative degree. Looking at the example a bit closer, one can realize that the reason for this is that the input v_1 appears in both output derivatives, while the input v_2 appears in none. So, an idea to fix this problem is to somehow make the input v_1 appear "later" (i.e., in higher-order derivatives of the outputs) and hope that the other input will catch up. How can we do this? Let us try using

$$\begin{aligned} v_1 &= \zeta \\ \dot{\zeta} &= u_1 \end{aligned}$$

This equation extends the state of the system to $(x, y, \theta, \phi, \zeta)$. The control vector fields are therefore $g_1 = (0, 0, 0, 0, 1)$ and $g_2 = (0, 0, 0, 1, 0)$. Note now that we have introduced a drift vector field $f = (\zeta\cos\theta, \zeta\sin\theta, \zeta\tan\phi/\ell, 0, 0)$. Now, the inputs do not appear in the first-order derivatives of the outputs,

$$\begin{aligned} \mathcal{L}_{g_1} h_1 &= 0, \quad \mathcal{L}_{g_2} h_1 = 0, \\ \mathcal{L}_{g_1} h_2 &= 0, \quad \mathcal{L}_{g_2} h_2 = 0. \end{aligned}$$

Let us see about the second-order derivatives. We compute

$$\mathcal{L}_f h_1 = \zeta \cos\theta$$
$$\mathcal{L}_f h_2 = \zeta \sin\theta$$

and then we have

$$\mathcal{L}_{g_1}\mathcal{L}_f h_1 = \cos\theta, \quad \mathcal{L}_{g_2}\mathcal{L}_f h_1 = 0$$
$$\mathcal{L}_{g_1}\mathcal{L}_f h_2 = \sin\theta, \quad \mathcal{L}_{g_2}\mathcal{L}_f h_2 = 0$$

and we again have the same problem, as before, i.e., the input u_1 appears earlier than the input v_2. Let us try with one more integrator

$$u_1 = \nu$$
$$\dot{\nu} = r_1$$

Therefore, the extended state is now $(x, y, \theta, \phi, \zeta, \nu)$, the inputs are r_1 and v_2, and the vector fields are $f = (\zeta\cos\theta, \zeta\sin\theta, \zeta\tan\phi/\ell, 0, \nu, 0)$, $g_1 = (0,0,0,0,0,1)$, and $g_2 = (0,0,0,1,0,0)$. Let us see if we finally got what we were looking for. The inputs do not appear in the first-order derivatives of the outputs

$$\mathcal{L}_{g_1} h_1 = 0, \quad \mathcal{L}_{g_2} h_1 = 0,$$
$$\mathcal{L}_{g_1} h_2 = 0, \quad \mathcal{L}_{g_2} h_2 = 0.$$

Additionally,

$$\mathcal{L}_f h_1 = \zeta \cos\theta$$
$$\mathcal{L}_f h_2 = \zeta \sin\theta$$

The inputs do not appear either in the second-order derivatives of the outputs

$$\mathcal{L}_{g_1}\mathcal{L}_f h_1 = 0, \quad \mathcal{L}_{g_2}\mathcal{L}_f h_1 = 0$$
$$\mathcal{L}_{g_1}\mathcal{L}_f h_2 = 0, \quad \mathcal{L}_{g_2}\mathcal{L}_f h_2 = 0$$

Additionally,

$$\mathcal{L}_f^2 h_1 = \nu \cos\theta - \zeta^2 \sin\theta \tan\phi/\ell$$
$$\mathcal{L}_f^2 h_2 = \nu \sin\theta + \zeta^2 \cos\theta \tan\phi/\ell$$

The third-order derivatives of the outputs now look like

$$\mathcal{L}_{g_1}\mathcal{L}_f^2 h_1 = \cos\theta, \quad \mathcal{L}_{g_2}\mathcal{L}_f^2 h_1 = -\zeta^2 \frac{\sin\theta}{\ell\cos^2\phi}$$
$$\mathcal{L}_{g_1}\mathcal{L}_f^2 h_2 = \sin\theta, \quad \mathcal{L}_{g_2}\mathcal{L}_f^2 h_2 = \zeta^2 \frac{\cos\theta}{\ell\cos^2\phi}$$

The determinant of the matrix

$$A = \begin{pmatrix} \mathcal{L}_{g_1}\mathcal{L}_f^2 h_1 & \mathcal{L}_{g_2}\mathcal{L}_f^2 h_1 \\ \mathcal{L}_{g_1}\mathcal{L}_f^2 h_2 & \mathcal{L}_{g_2}\mathcal{L}_f^2 h_2 \end{pmatrix}$$

is

$$\det(A) = \zeta^2 \frac{1}{\ell\cos^2\phi}$$

Therefore, so long as $\zeta \neq 0$ and $\phi \neq \pm\pi/2, 3\pi/2$, the matrix A is invertible, and the system has vector relative degree $(3,3)$. Note that $\rho = 3+3 = 6$, which is the dimension of the extended state $(x, y, \theta, \phi, \zeta, \nu)$. So, by extending the state of the system and introducing carefully-chosen integrators, we are able to fully linearize the system. Doing control design, e.g., trajectory tracking, is now reasonably straightforward. Note that we have fully linearized the system not through a static state feedback controller, but through a dynamic state feedback controller. The question is, can this be done in general? Can we write our procedure in general terms? This is what we address next.

Achieving relative degree via dynamic extension

The purpose of this section is to show that, under certain assumptions, it is possible to modify a system which does not have a vector relative degree into a new system which does. This cannot be achieved by static state feedback. Instead, we use a dynamic state feedback of the form

$$\begin{aligned} u &= \alpha(x,\zeta) + \beta(x,\zeta)v \\ \dot{\zeta} &= \gamma(x,\zeta) + \delta(x,\zeta)v \end{aligned}$$

This dynamic feedback is said to be a *regularizing dynamic extension* for (1) if the composite system has a vector relative degree.

The basic idea of the dynamic extension algorithm is to add integrators at the appropriate places to delay the appearances of inputs and make the system have a vector relative degree. The following formulation of the dynamic extension algorithm is iterative. It tries to take care of problems one step at a time.

Dynamic extension algorithm: Suppose that the matrix $A(x)$ in (2) has constant rank $p < m$. Let a_i, $i \in \{1, \ldots, m\}$ denote the ith row of $A(x)$. Without loss of generality (rearranging the order of the output channels if necessary), it is possible to find smooth functions $c_1, \ldots, c_p$ such that

$$a_{p+1}(x) = \sum_{i=1}^{p} c_i(x) a_i(x).$$

> % *This fact is not good for achieving a relative degree. Somehow, some input has appeared a bit too soon when deriving one of the first p outputs*

It must be the case that there exists $i_0 \in \{1, \ldots, p\}$ and $j_0 \in \{1, \ldots, m\}$ such that

$$a_{i_0 j_0} = \mathcal{L}_{g_{j_0}} \mathcal{L}_f^{\rho_{i_0}-1} h_{i_0} \neq 0.$$

> % *The input j_0 is the culprit. Let us make it show up later in the derivatives of the output i_0 using an integrator.*

Let us then define

$$u_j = v_j, \quad j \neq j_0 \tag{5a}$$

$$u_{j_0} = \frac{1}{a_{i_0 j_0}}\big(- \mathcal{L}_f^{\rho_{i_0}} h_{i_0} + \zeta - \sum_{\substack{j=1 \\ j \neq j_0}}^{m} a_{i_0 j} v_j \big) \tag{5b}$$

$$\dot{\zeta} = v_{j_0} \tag{5c}$$

% *What do we achieve with this? Basically, we delay the appearance of the input j_0 when making derivatives of the output h_{i_0}.*

With this choice, we now have

$$\begin{aligned} y_{i_0}^{(\rho_{i_0})} &= \mathcal{L}_f^{\rho_{i_0}} h_{i_0} + \sum_{j=1}^{m} \mathcal{L}_{g_j} \mathcal{L}_f^{\rho_{i_0}-1} h_{i_0} u_j \\ &= \mathcal{L}_f^{\rho_{i_0}} h_{i_0} + a_{i_0 j_0} u_{j_0} + \sum_{\substack{j=1 \\ j \neq j_0}}^{m} a_{i_0 j} u_j = \zeta \end{aligned}$$

Therefore, in the system with state (x, ζ), the first time an input appears in the derivatives of the output h_{j_0} is when we do the $\rho_{i_0} + 1$ derivative,

$$y_{i_0}^{(\rho_{i_0}+1)} = \dot{\zeta} = v_{j_0}$$

% *So we have accomplished something: now the input j_0 appears "one derivative later." The hope is that this will make the resulting matrix A invertible. Otherwise, we repeat the process.*

So, as a result of the addition of the integrator, the system we are dealing with now has state (x, ζ) of dimension $n+1$,

$$\dot{x} = f(x) + \sum_{\substack{j=1 \\ j \neq j_0}} v_j g_j(x) + g_{j_0}(x) \frac{1}{a_{i_0 j_0}} \Big(- \mathcal{L}_f^{\rho_{i_0}} h_{i_0} + \zeta - \sum_{\substack{j=1 \\ j \neq j_0}}^{m} a_{i_0 j} v_j \Big) \tag{6a}$$

$$\dot{\zeta} = v_{j_0} \tag{6b}$$

$$y_1 = h_1(x) \tag{6c}$$

$$\vdots \tag{6d}$$

$$y_m = h_m(x) \tag{6e}$$

If this system has a relative degree, then the algorithm terminates. Otherwise, we repeat the procedure starting with (6).

In general, we have no guarantee that the dynamic extension algorithm will succeed. However, if it does, then the regularizing dynamic feedback it generates has necessarily the least possible dimension. This fact is a consequence of the following result.

Proposition *Suppose the dynamic extension algorithm has been iterated k times. Let*

$$u = H(x,\zeta) + K(x,\zeta)\tilde{v} \tag{7a}$$
$$\dot{\zeta} = F(x,\zeta) + G(x,\zeta)\tilde{v} \tag{7b}$$

with $\zeta \in \mathbb{R}^k$ denote the composition of the k feedback laws of the form (5) constructed at each stage of the algorithm. On the other hand, assume there exists a regularizing dynamic extension

$$u = H(x,\nu) + K(x,\nu)v \tag{8a}$$
$$\dot{\nu} = F(x,\nu) + G(x,\nu)v \tag{8b}$$

with $\nu \in \mathbb{R}^\ell$. Then $k \leq \ell$, and the feedback (8) is the result of the composition of the feedback (7) with an additional regularizing dynamic extension of the form

$$\tilde{v} = \tilde{\alpha}(x,\zeta,z) + \tilde{\beta}(x,\zeta,z)v$$
$$\dot{z} = \tilde{\gamma}(x,\zeta,z) + \tilde{\delta}(x,\zeta,z)v$$

with $z \in \mathbb{R}^{\ell-k}$.

Interestingly enough, it is currently an open problem to come up with sufficient and necessary conditions that guarantee that a system can be linearized by means of dynamic feedback. In other words, an equivalent of Theorem 2.9 is not known for the dynamic feedback case.

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