Linear and Nonlinear Optimization

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Necessary conditions for optimal input

The problem is to find an optimal input (u^*) that causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{1}$$

to follow a trajectory (x^*) that minimizes the performance measure

$$J(\mathbf{x}(t), t) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau. \tag{2}$$

Necessary conditions for optimal input

The Hamiltonian is given by

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^{\mathrm{T}}(t) \left[\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)\right]. \tag{3}$$

The necessary conditions of optimality are

$$\dot{\mathbf{x}}^*(t) = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t), \tag{4}$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t), \tag{5}$$

$$0 = \frac{\partial \mathcal{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t). \tag{6}$$

The system

$$\dot{x}_1(t) = x_2(t) \quad (7)$$
$$\dot{x}_2(t) = -x_2(t) + u(t) \quad (8)$$

is to be controlled so that its input is conserved. Therefore, the performance measure is given by

$$J(u) = \int_{t_0}^{t_f} \frac{1}{2} u^2(t) dt. \quad (9)$$

The Hamiltonian (3) is given by

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = \frac{1}{2} u^2(t) + p_1(t) x_2(t) - p_2(t) x_2(t) + p_2(t) u(t). \quad (10)$$

Example

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = \frac{1}{2}u^2(t) + p_1(t)x_2(t) - p_2(t)x_2(t) + p_2(t)u(t). \tag{11}$$

The necessary conditions for optimality are

$$\dot{p}_1^*(t) = \frac{\partial \mathcal{H}}{\partial x_1} = 0 \tag{12}$$

$$\dot{p}_2^*(t) = -\frac{\partial \mathcal{H}}{\partial x_2} = -p_1^*(t) + p_2^*(t), \tag{13}$$

and

$$0 = \frac{\partial \mathcal{H}}{\partial u} = u^*(t) + p_2^*(t). \tag{14}$$

The plant is described by the linear state equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{15}$$

which may have time-varying coefficients. The performance measure to be minimized is

$$J = \frac{1}{2}\mathbf{x}^{\mathrm{T}}(t_f)\mathbf{H}\mathbf{x}(t_f) + \int_{t_0}^{t_f} \frac{1}{2}\left[\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u}\right] dt, \tag{16}$$

where the final time t_f is fixed. Further, **H** and **Q** are real symmetric positive semi-definite matrices. Finally, **R** is a real symmetric positive definite matrix.
The Hamiltonian is

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^{\mathrm{T}}(t)\left[\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)\right]. \tag{17}$$

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u} + \mathbf{p}^{\mathrm{T}}\mathbf{A}(t)\mathbf{x}(t) + \mathbf{p}^{\mathrm{T}}\mathbf{B}(t)\mathbf{u}(t), \tag{18}$$

and the necessary conditions for optimality are

$$\dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) + \mathbf{B}(t)\mathbf{u}^*(t), \tag{19}$$

$$\dot{\mathbf{p}}^*(t) = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}^{\mathrm{T}}(t)\mathbf{p}^*(t), \tag{20}$$

$$0 = \mathbf{R}(t)\mathbf{u}^*(t) + \mathbf{B}^{\mathrm{T}}(t)\mathbf{p}^*(t). \tag{21}$$

Solving (21) for $\mathbf{u}^*(t)$ yields

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{p}^*(t). \tag{22}$$

Substitution of (22) into (19) yields

$$\dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{p}^*(t). \tag{23}$$

Putting (20) and (23) into the following matrix format

$$\begin{bmatrix} \dot{\mathbf{x}}^*(t) \\ -- \\ \dot{\mathbf{p}}^*(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(t) & | & -\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t) \\ -- & -- & -- \\ -\mathbf{Q}(t) & | & \mathbf{p}^*(t) \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(t) \\ -- \\ \mathbf{p}^*(t) \end{bmatrix}. \quad (24)$$

The solution of these equations has the following form

$$\begin{bmatrix} \mathbf{x}^*(t_f) \\ -- \\ \mathbf{p}^*(t_f) \end{bmatrix} = \varphi(t_f, t) \begin{bmatrix} \mathbf{x}^*(t) \\ -- \\ \mathbf{p}^*(t) \end{bmatrix}, \quad (25)$$

where $\varphi(t_f, t)$ is the state-transition matrix of the system (24).

$$\begin{bmatrix} \mathbf{x}^*(t_f) \\ -- \\ \mathbf{p}^*(t_f) \end{bmatrix} = \begin{bmatrix} \varphi_{11}(t_f, t) & | & \varphi_{12}(t_f, t) \\ -- & -- & -- \\ \varphi_{21}(t_f, t) & | & \varphi_{22}(t_f, t) \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(t) \\ -- \\ \mathbf{p}^*(t) \end{bmatrix}, \quad (26)$$

From the boundary condition, the final co-states are related to the final states using

$$\mathbf{p}^*(t_f) = \mathbf{H}\mathbf{x}^*(t_f). \tag{27}$$

Solving for $\mathbf{p}^*(t_f)$, we obtain

$$\mathbf{p}^*(t) = \left[\varphi_{22}(t_f, t) - \mathbf{H}\varphi_{12}(t_f, t)\right]^{-1} \left[\mathbf{H}\varphi_{11}(t_f, t) - \varphi_{21}(t_f, t)\right] \mathbf{x}^*(t). \tag{28}$$

$$\mathbf{p}^*(t) = \mathbf{K}(t)\mathbf{x}^*(t). \tag{29}$$

The optimal input is given by

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{K}(t)\mathbf{x}^*(t). \tag{30}$$

Example

It is desired to determine the input (using the principle of optimality and the Hamilton-Jacobi-Bellman equation) that causes the plant

$$\dot{x}_1 = x_2(t) \quad (31)$$
$$\dot{x}_2 = -x_1(t) - 2x_2(t) + u(t) \quad (32)$$

to minimize the performance measure

$$J = 10x_1^2(T) + \frac{1}{2}\int_0^T \left[x_1^2(t) + 2x_2^2(t) + u^2(t)\right]. \quad (33)$$

Thanks

Questions please