Assignment 1 - Deadline 1 week

Problem 1
This exercise is designed to show how small changes in the coefficients of a linear system can affect a fixed point that is a center. Consider the linear system
\begin{align*}
\dot{x}_1 &= 0x_1 + x_2 \\
\dot{x}_2 &= -x_1 + 0x_2
\end{align*}
(1)
(2)

- Find the eigenvalues of the coefficient matrix, classify the fixed point (which is the origin), and determine its stability.

Now consider the linear system
\begin{align*}
\dot{x}_1 &= \varepsilon x_1 + x_2 \\
\dot{x}_2 &= -x_1 + \varepsilon x_2
\end{align*}
(3)
(4)
where $\varepsilon$ is arbitrarily small.

- Find the eigenvalues of the coefficient matrix. Show that no matter how small $|\varepsilon| \neq 0$ is, the center has been changed into a different type of fixed point. What type of fixed point is it?
- Determine the stability of the fixed point for $\varepsilon < 0$ and $\varepsilon > 0$.

Problem 2
This exercise is designed to show how small changes in the coefficients of a linear system can affect the nature of a fixed point when the eigenvalues of the coefficient matrix are equal. Consider the linear system
\begin{align*}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= 0x_1 - x_2
\end{align*}
(5)
(6)

- Find the eigenvalues of the coefficient matrix, classify the fixed point (which is the origin), and determine its stability.

Now consider the linear system
\begin{align*}
\dot{x}_1 &= \varepsilon x_1 + x_2 \\
\dot{x}_2 &= -\varepsilon x_1 + \varepsilon x_2
\end{align*}
(7)
(8)
where $\varepsilon$ is arbitrarily small.

- Find the eigenvalues of the coefficient matrix.
- Classify the fixed point and determine its stability if $\varepsilon > 0$.
- Classify the fixed point and determine its stability if $\varepsilon < 0$ but $|\varepsilon| \neq 0$.

Problem 3
For each of the following systems, (i) find the equilibrium points, (ii) classify their type and stability, (iii) then sketch the nullclines, the vector field, and a plausible phase portrait.
1. \( \dot{x}_1 = x_1 - x_2 \), \( \dot{x}_2 = 1 - \exp^{x_1} \)
2. \( \dot{x}_1 = x_1 - x_1^3 \), \( \dot{x}_2 = -x_2 \)
3. \( \dot{x}_1 = (x_1 - x_2) \), \( \dot{x}_2 = 2x_1 - x_2 \)
4. \( \dot{x}_1 = x_2 \), \( \dot{x}_2 = x_1 (1 + x_2) - 1 \)
5. \( \dot{x}_1 = (2 - x_1 - x_2) \), \( \dot{x}_2 = x_1 - x_2 \)
6. \( \dot{x}_1 = x_1^2 - x_2 \), \( \dot{x}_2 = x_1 - x_2 \)

**Problem 4**

Determine the type and stability of the fixed point at the origin for the nonlinear system

\[
\begin{align*}
\dot{x}_1 &= ax_1^3 - x_2 \\
\dot{x}_2 &= x_1 + ax_2^3
\end{align*}
\]  
(9)  
(10)

for all real values of the parameter \( a \).

**Problem 5**

For each of the following systems, (i) find the equilibrium points, (ii) classify their type and stability, (iii) then sketch the nullclines, the vector field, and a plausible phase portrait

1. \( \dot{x}_1 = x_1 - x_2 + x_1 x_2 \), \( \dot{x}_2 = 3x_1 - 2x_2 - x_1 x_2 \)
2. \( \dot{x}_1 = x_1 + x_1^2 + x_2^2 \), \( \dot{x}_2 = x_2 - x_1 x_2 \)
3. \( \dot{x}_1 = -2x_1 - x_2 - x_1(x_1^2 + x_2^2) \), \( \dot{x}_2 = x_1 - x_2 + x_2(x_1^2 + x_2^2) \)
4. \( \dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2) \), \( \dot{x}_2 = -x_1 - x_2 + x_2(1 - x_1^2 - x_2^2) \)
5. \( \dot{x}_1 = 2x_1 + x_2 + x_1 x_2^3 \), \( \dot{x}_2 = x_1 - 2x_2 - x_1 x_2 \)
6. \( \dot{x}_1 = x_1 + 2x_1^2 - x_2^2 \), \( \dot{x}_2 = x_1 - 2x_2 + x_1^3 \)
7. \( \dot{x}_1 = x_2 \), \( \dot{x}_2 = x_2 - x_1 + \mu x_2(1 - x_1^2) \), \( \mu > 0 \)
8. \( \dot{x}_1 = 1 + x_1 - \exp^{-x_1} \), \( \dot{x}_2 = x_2 - \sin x_1 \)
9. \( \dot{x}_1 = (1 + x_1) \sin x_2 \), \( \dot{x}_2 = 1 - x_1 - \cos x_2 \)
10. \( \dot{x}_1 = \exp(-x_1 + x_2) - \cos x_1 \), \( \dot{x}_2 = \sin(x_1 - 3x_2) \)