Robotics

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A vector $\mathbf{v}$ in a reference frame A depends on a scalar verifiable q. We can say that $\mathbf{v}$ is a vector function of q in A. Otherwise, $\mathbf{v}$ is independent of q in A.

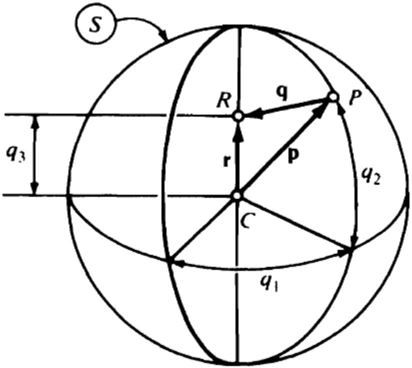


Figure: P is dependent on q_1 and q_2 in S and independent on q_3 in S.

A vector **v** may be a function of a variable q in one reference frame, but be independent of q in another reference frame.

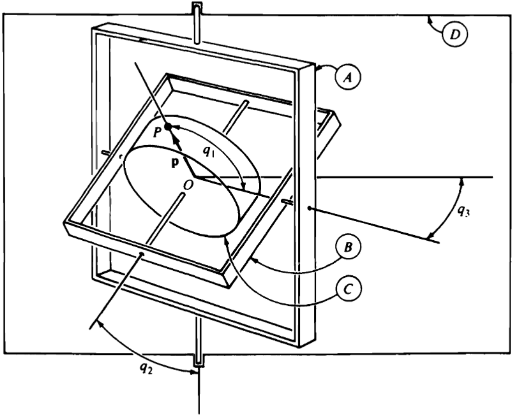


Figure: A gyroscope.

P is the position vector from point O to point P of C. P is a function of q_1 both in A and B, but is independent of q_1 in C.

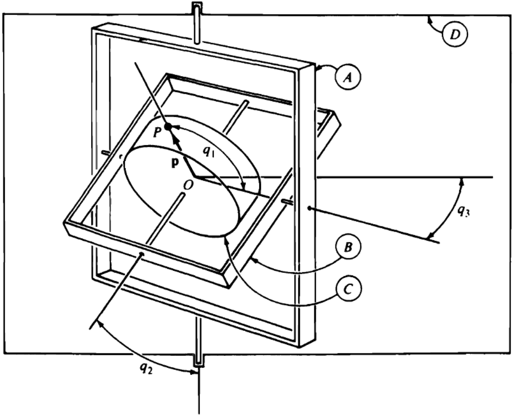


Figure: A gyroscope.

P is a function of q_2 both A, but is independent of q_2 in B and in C.

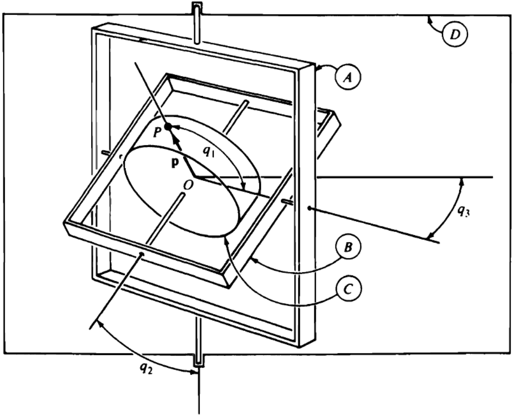


Figure: A gyroscope.

P is independent of q_3 both A, B, and C, but is a function of q_3 in D.

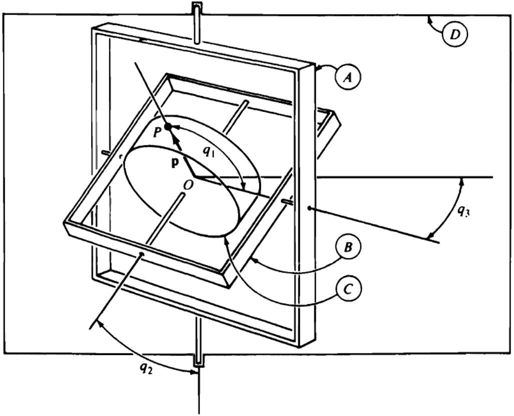


Figure: A gyroscope.

Vector Functions

In a reference frame A, a vector function $\mathbf{v}$ of n scalar variables $q_1, \ldots, q_n$, let $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ be a set of nonparallel, noncoplanar unit vectors fixed in A. Then there exist three unique scalar functions v_1, v_2, and v_3 of $q_1, \ldots, q_n$ such that

$$\mathbf{v} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3 \tag{1}$$

When $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are mutually perpendicular unit vectors, then it follows from (1) that the v_i is given by

$$v_i = \mathbf{v} \cdot \mathbf{a}_i \tag{2}$$

Therefore, (1) can be rewritten as follow:

$$\mathbf{v} = \mathbf{v} \cdot \mathbf{a}_1\mathbf{a}_1 + \mathbf{v} \cdot \mathbf{a}_2\mathbf{a}_2 + \mathbf{v} \cdot \mathbf{a}_3\mathbf{a}_3 \tag{3}$$

Vector Functions

$\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are mutualy perpendicular unit vectors fixed in A and B, respectively. Vector $\mathbf{p}$ can be expressed both as

$$\mathbf{p} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3 \tag{4}$$

and as

$$\mathbf{p} = \beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + \beta_3 \mathbf{b}_3 \tag{5}$$

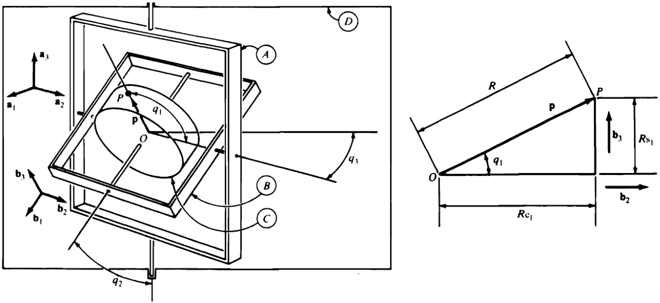


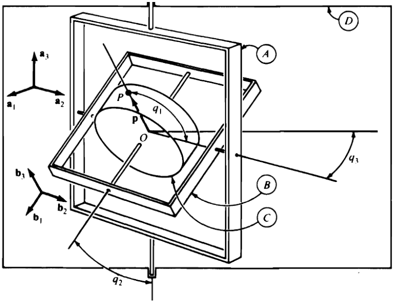
Figure: A gyroscope. α_i and β_i for $i = 1, 2, 3$ are functions of q_1, q_2, and q_3.

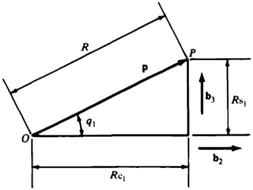
To determine the functions α_i and β_i, note that, if C has a radius R, one can proceed from O to P by moving through the distances $R\cos q_1$ and $R\sin q_1$ in the directions of $\mathbf{b}_2$ and $\mathbf{b}_3$, respectively, therefore

$$\mathbf{p} = R\cos q_1\mathbf{b}_2 + R\sin q_1\mathbf{b}_3 \qquad (6)$$

Comparing (5) to (6), we find that

$$\beta_1 = 0 \qquad \beta_2 = R\cos q_1 \qquad \beta_3 = R\sin q_1 \qquad (7)$$





Based on (4), one can write

$$\alpha_1 = \mathbf{p}\cdot\mathbf{a}_1 = R\left(\cos q_1 \mathbf{b}_2\cdot\mathbf{a}_1 + \sin q_1 \mathbf{b}_3\cdot\mathbf{a}_1\right) \quad (8)$$
$$\alpha_2 = \mathbf{p}\cdot\mathbf{a}_2 = R\left(\cos q_1 \mathbf{b}_2\cdot\mathbf{a}_2 + \sin q_1 \mathbf{b}_3\cdot\mathbf{a}_2\right) \quad (9)$$
$$\alpha_3 = \mathbf{p}\cdot\mathbf{a}_3 = R\left(\cos q_1 \mathbf{b}_2\cdot\mathbf{a}_3 + \sin q_1 \mathbf{b}_3\cdot\mathbf{a}_3\right) \quad (10)$$

We also know that

$$\mathbf{b}_2\cdot\mathbf{a}_1 = 0 \qquad \mathbf{b}_2\cdot\mathbf{a}_2 = 1 \qquad \mathbf{b}_2\cdot\mathbf{a}_3 = 0 \quad (11)$$

$$\mathbf{b}_3\cdot\mathbf{a}_1 = \cos q_2 \qquad \mathbf{b}_3\cdot\mathbf{a}_2 = 0 \qquad \mathbf{b}_3\cdot\mathbf{a}_3 = \sin q_2 \quad (12)$$

Therefore, the functions α_i of $\mathbf{p}$ are

$$\alpha_1 = R\sin q_1 \cos q_2 \qquad \alpha_2 = R\cos q_1 \qquad \alpha_3 = R\sin q_1 \sin q_2 \quad (13)$$

First Derivatives

If $\mathbf{v}$ is a vector function of n scalar variables $q_1, \ldots, q_n$ in a reference frame A, then n vectors called first partial derivatives of $\mathbf{v}$ in A and denoted by the symbols

$$ {}^A\frac{\partial \mathbf{v}}{\partial q_r} \quad \text{or} \quad {}^A\frac{\partial}{\partial q_r}(\mathbf{v}), \qquad (r = 1, \ldots, n) $$

are defined as follows: Let $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ be any any nonparallel, noncoplanar unit vectors fixed in A, and let v_i be the $\mathbf{a}_i$ measure number of v then

$$ {}^A\frac{\partial \mathbf{v}}{\partial q_r} \triangleq \sum_{i=1}^{3} \frac{\partial v_i}{\partial q_r}\mathbf{a}_i, \qquad (r = 1, \ldots, n) \tag{14} $$

When $\mathbf{v}$ is regarded as a vector function of only a single scalar variable in $A$for instance the time $t$then this definition reduces to that of the ordinary derivative of $\mathbf{v}$ with respect to t in A, to

$$ {}^A\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q_r} \triangleq \sum_{i=1}^{3} \frac{\mathrm{d}v_i}{\mathrm{d}q_r}\mathbf{a}_i, \qquad (r = 1, \ldots, n) \tag{15} $$

First Derivatives

The vector **p** possesses partial derivatives with respect to q_1, q_2, and q_3 in each of the reference frames A, B, and C. To form ${}^A\frac{\partial \mathbf{p}}{\partial q_r}$, $(r = 1, \ldots, n)$ we use (14) as follows:

$$ {}^A\frac{\partial \mathbf{p}}{\partial q_r} = \left[\frac{\partial}{\partial q_r} R \sin q_1 \cos q_2\right] \mathbf{a}_1 + \left[\frac{\partial}{\partial q_r} R \cos q_1\right] \mathbf{a}_2 + \left[\frac{\partial}{\partial q_r} R \sin q_1 \sin q_2\right] \mathbf{a}_3 \qquad (16) $$

Consequently,

$$ {}^A\frac{\partial \mathbf{p}}{\partial q_1} = R\left(\cos q_1 \cos q_2 \mathbf{a}_1 - \sin q_1 \mathbf{a}_2 + \cos q_1 \sin q_2 \mathbf{a}_3\right) \qquad (17) $$

$$ {}^A\frac{\partial \mathbf{p}}{\partial q_2} = R \sin q_1 \left(-\sin q_2 \mathbf{a}_1 + \cos q_2 \mathbf{a}_3\right) \qquad (18) $$

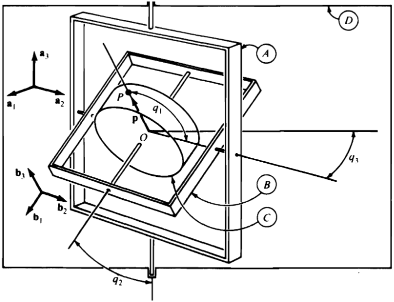
$$ {}^A\frac{\partial \mathbf{p}}{\partial q_3} = 0 \qquad (19) $$

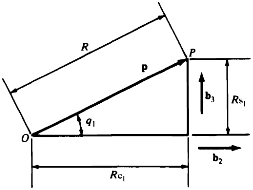
p is independent of q_3 in A.

First Derivatives

$$
\begin{aligned}
{}^{A}\frac{\partial \mathbf{p}}{\partial q_1} &= R\left(\cos q_1 \cos q_2 \mathbf{a}_1 - \sin q_1 \mathbf{a}_2 + \cos q_1 \sin q_2 \mathbf{a}_3\right) && (20)\\
{}^{A}\frac{\partial \mathbf{p}}{\partial q_2} &= R \sin q_1 \left(-\sin q_2 \mathbf{a}_1 + \cos q_2 \mathbf{a}_3\right) && (21)\\
{}^{A}\frac{\partial \mathbf{p}}{\partial q_3} &= 0 && (22)
\end{aligned}
$$

$\mathbf{p}$ is independent of q_3 in A.





First Derivatives

Proceeding similarly to determine ${}^B\frac{\partial \mathbf{p}}{\partial q_r}$, $(r = 1, \ldots, n)$, we obtain

$$ {}^B\frac{\partial \mathbf{p}}{\partial q_1} = R\left(-\sin q_1 \mathbf{b}_2 + \cos q_1 \mathbf{b}_3\right) \qquad {}^B\frac{\partial \mathbf{p}}{\partial q_2} = 0 \qquad {}^B\frac{\partial \mathbf{p}}{\partial q_3} = 0 \quad (23) $$

$\mathbf{p}$ is independent of q_2 and q_3 in B.

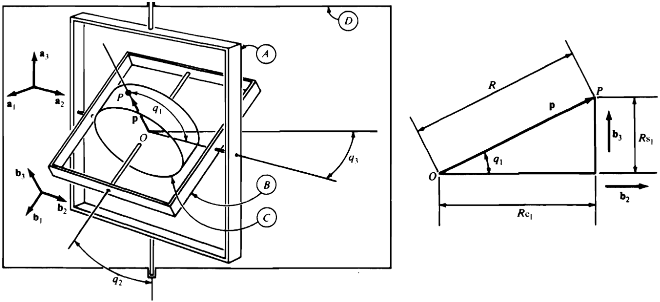


Figure: A gyroscope.

Similarly to determine ${}^{C}\frac{\partial \mathbf{p}}{\partial q_r}$, $(r = 1, \ldots, n)$, we obtain

$$ {}^{C}\frac{\partial \mathbf{p}}{\partial q_r} = 0 \qquad (24) $$

$\mathbf{p}$ is independent of q_1, q_2 and q_3 in C.

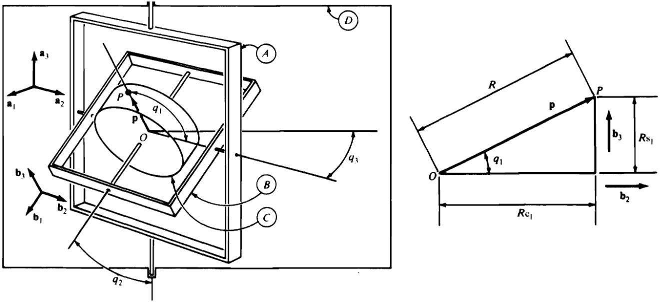


Figure: A gyroscope.

First Derivatives

Suppose now that q_i are specified as explicit functions of time t

$$q_1 = t \qquad q_2 = 2t \qquad q_3 = 3t$$

The α_i can be expressed as

$$\alpha_1 = R \sin t \cos 2t \qquad \alpha_2 = R \cos t \qquad \alpha_3 = R \sin t \sin 2t$$

and the ordinary derivative of $\mathbf{p}$ with respect to t in A is seen to be given by

$$\begin{aligned} {}_A\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= \frac{\mathrm{d}\alpha_1}{\mathrm{d}t}\mathbf{a}_1 + \frac{\mathrm{d}\alpha_2}{\mathrm{d}t}\mathbf{a}_2 + \frac{\mathrm{d}\alpha_3}{\mathrm{d}t}\mathbf{a}_3 \\ &= R[(\cos t \cos 2t - 2 \sin t \sin 2t)\mathbf{a}_1 \\ &-(\sin t)\mathbf{a}_2 + (\cos t \sin 2t + 2 \sin t \cos 2t)\mathbf{a}_3] \end{aligned} \tag{25}$$

First Derivatives

while the ordinary derivative of **p** with respect to t in B is

$$ {}^{B}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = R\,(-\sin t\mathbf{b}_2 + \cos t\mathbf{b}_3) \tag{26} $$

Finally,

$$ {}^{C}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = 0 \tag{27} $$

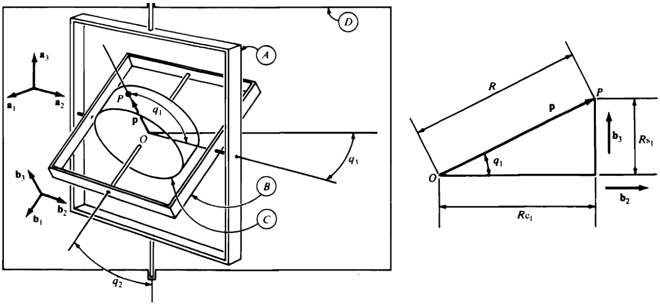


Figure: A gyroscope.

Thank You!
Questions please