Robotics

Islam S. M. Khalil

German University in Cairo

September 19, 2016

Angular Velocity

Let $\mathbf{b}_l$, $\mathbf{b}_2$, and $\mathbf{b}_3$ form a right-handed set of mutually perpendicular unit vectors fixed in a rigid body B moving in a reference frame A. The angular velocity of B in A, denoted by ${}^{A}\omega^{B}$, is defined as

$$ {}^{A}\omega^{B} \triangleq \mathbf{b}_1 \frac{{}^{A}\mathrm{d}\mathbf{b}_2}{\mathrm{d}t} \cdot \mathbf{b}_3 + \mathbf{b}_2 \frac{{}^{A}\mathrm{d}\mathbf{b}_3}{\mathrm{d}t} \cdot \mathbf{b}_1 + \mathbf{b}_3 \frac{{}^{A}\mathrm{d}\mathbf{b}_1}{\mathrm{d}t} \cdot \mathbf{b}_2 \quad (1) $$

One task facilitated by the use of angular velocity vectors is the time-differentiation of vectors fixed in a rigid body, for it enables one to obtain the first time-derivative of such a vector by performing a cross-multiplication. Specifically, if β is any vector fixed in B, then

$$ \frac{{}^{A}\mathrm{d}\beta}{\mathrm{d}t} = {}^{A}\omega^{B} \times \beta \quad (2) $$

$$^{A}\omega^{B} \triangleq \mathbf{b}_1 \frac{^{A}\mathrm{d}\mathbf{b}_2}{\mathrm{d}t} \cdot \mathbf{b}_3 + \mathbf{b}_2 \frac{^{A}\mathrm{d}\mathbf{b}_3}{\mathrm{d}t} \cdot \mathbf{b}_1 + \mathbf{b}_3 \frac{^{A}\mathrm{d}\mathbf{b}_1}{\mathrm{d}t} \cdot \mathbf{b}_2 \quad (3)$$

$$\frac{^{A}\mathrm{d}\beta}{\mathrm{d}t} = {}^{A}\omega^{B} \times \beta \quad (4)$$

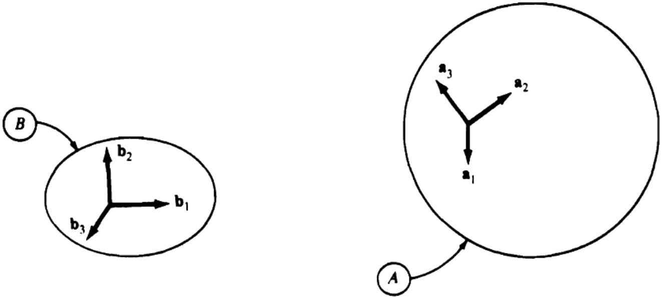


Figure: Angular velocity of *B* with respect to *A*.

Derivation Using dots to denote time-differentiation in A, one can rewrite (1)

$$ {}^{A}\omega^{B} \triangleq \mathbf{b}_1\dot{\mathbf{b}}_2 \cdot \mathbf{b}_3 + \mathbf{b}_2\dot{\mathbf{b}}_3 \cdot \mathbf{b}_1 + \mathbf{b}_3\dot{\mathbf{b}}_1 \cdot \mathbf{b}_2 \tag{5} $$

and cross-multiplication of (5) with $\mathbf{b}_1$ gives

$$ {}^{A}\omega^{B} \times \mathbf{b}_1 = \mathbf{b}_2 \times \mathbf{b}_1\dot{\mathbf{b}}_3 \cdot \mathbf{b}_1 + \mathbf{b}_3 \times \mathbf{b}_1\dot{\mathbf{b}}_1 \cdot \mathbf{b}_2 \tag{6} $$

We also know that

$$ \mathbf{b}_2 = \mathbf{b}_3 \times \mathbf{b}_1 \qquad\qquad \mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2 \tag{7} $$

Substitution of (7) in (6) yields

$$ {}^{A}\omega^{B} \times \mathbf{b}_1 = -\mathbf{b}_3\dot{\mathbf{b}}_3 \cdot \mathbf{b}_1 + \mathbf{b}_2\dot{\mathbf{b}}_1 \cdot \mathbf{b}_2 \tag{8} $$

Derivation Moreover, time-differentiation of the equations $\mathbf{b}_1 \cdot \mathbf{b}_1 = 1$ and $\mathbf{b}_3 \cdot \mathbf{b}_1 = 0$ yields

$$\dot{\mathbf{b}}_1 \cdot \mathbf{b}_1 \qquad \dot{\mathbf{b}}_3 \cdot \mathbf{b}_1 = -\dot{\mathbf{b}}_1 \cdot \mathbf{b}_3 \tag{9}$$

rewrite (8)

$$^A\omega^B \times \mathbf{b}_1 = \mathbf{b}_1 \dot{\mathbf{b}}_1 \cdot \mathbf{b}_1 + \mathbf{b}_2 \dot{\mathbf{b}}_1 \cdot \mathbf{b}_2 + \mathbf{b}_3 \dot{\mathbf{b}}_1 \cdot \mathbf{b}_3 \tag{10}$$

But the right-hand member of this equation is simply a way of writing $\dot{\mathbf{b}}_1$.

$$^A\omega^B \times \mathbf{b}_1 = \dot{\mathbf{b}}_1 \tag{11}$$

Similarly,

$$^A\omega^B \times \mathbf{b}_2 = \dot{\mathbf{b}}_2 \qquad ^A\omega^B \times \mathbf{b}_3 = \dot{\mathbf{b}}_3 \tag{12}$$

and, after expressing any vector β fixed in B as

$$\beta = \beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + \beta_3 \mathbf{b}_3 \tag{13}$$

Derivation where β_1, β_2, and β_3 are constants, so that

$$\dot{\beta} = \beta_1 \dot{\mathbf{b}}_1 + \beta_2 \dot{\mathbf{b}}_2 + \beta_3 \dot{\mathbf{b}}_3 \tag{14}$$

Therefore,

$$\dot{\beta} = \beta_1 \ {}^{A}\omega^{B} \times \mathbf{b}_1 + \beta_2 \ {}^{A}\omega^{B} \times \mathbf{b}_2 + \beta_3 \ {}^{A}\omega^{B} \times \mathbf{b}_3 \tag{15}$$

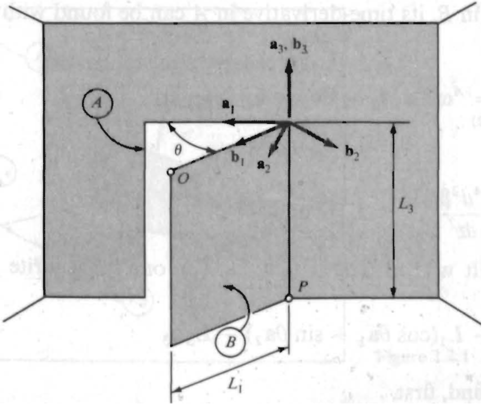
Finally we obtain

$$\dot{\beta} = \ {}^{A}\omega^{B} \times (\beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + \beta_3 \mathbf{b}_3) = \ {}^{A}\omega^{B} \times \beta \tag{16}$$

Auxiliary Reference Frames

$$\dot{\beta} = \ {}^{A}\omega^{B} \times \beta \tag{17}$$

Example. Here B represents a door supported by hinges in a room A. Mutually perpendicular unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are fixed in A, with $\mathbf{a}_3$ parallel to the axis of the hinges, and mutually perpendicular unit vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are fixed in B, with $\mathbf{b}_3 = \mathbf{a}_3$.



Angular Velocity

If θ is the radian measure of the angle between $\mathbf{a}_1$ and $\mathbf{b}_1$ as shown below, then $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ and $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are related to each other as indicated in the table below

	$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$
$\mathbf{a}_1$	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_2$	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_3$	0	0	1

The angular velocity of B in A, ${}^A\omega^B$ is given by

$$ {}^A\omega^B = \dot{\theta}\mathbf{b}_3 \tag{18} $$

The second time-derivative in A of the position vector from the point O to the point P, that is, of the vector β given by

$$ \beta = -L_1\mathbf{b}_1 - L_3\mathbf{b}_3 \tag{19} $$

The derivative of β in A is given by

$$ \frac{{}^A\partial\beta}{\partial t} = \dot{\theta}\mathbf{b}_3 \times (-L_1\mathbf{b}_1 - L_3\mathbf{b}_3) = -L_1\dot{\theta}\mathbf{b}_2 \tag{20} $$

$$\frac{{}^{A}\partial^{2}\beta}{\partial t^{2}} = -L_{1}\ddot{\theta}\mathbf{b}_{2} - L_{1}\dot{\theta}\frac{{}^{A}\partial\mathbf{b_2}}{\partial t} \quad (21)$$

Since $\mathbf{b}_2$ is a vector fixed in B, its time-derivative in A can be found as follows:

$$\frac{{}^{A}\partial\mathbf{b_2}}{\partial t} = {}^{A}\omega^{B} \times \mathbf{b}_{2} = -\dot{\theta}\mathbf{b}_{1} \quad (22)$$

Therefore,

$$\frac{{}^{A}\partial^{2}\beta}{\partial t^{2}} = L_{1}\left(\dot{\theta}^{2}\mathbf{b}_{1} - \ddot{\theta}\mathbf{b}_{2}\right) \quad (23)$$

In more complex situations, that is, when the motion of B in A is more complicated than that of a door B in a room A, the use of the angular velocity vector as an "operator" which, through cross-multiplication, produces time-derivatives, is all the more advantageous.

Differentiation in Two Reference Frames

If A and B are any two reference frames, the first time-derivatives of any vector $\mathbf{v}$ in A and in B are related to each other as follows:

$$\frac{{}^{A}\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{{}^{B}\mathrm{d}\mathbf{v}}{\mathrm{d}t} + {}^{A}\omega^{B} \times \mathbf{v} \tag{24}$$

where ${}^{A}\omega^{B}$ is the angular velocity of B in A.

Derivation. Let us define a vector $\mathbf{v}$ in B so that

$$\mathbf{v} = \sum_{i=1}^{3} v_i \mathbf{b}_i \tag{25}$$

The time-derivative of $\mathbf{v}$ in A is

$$\frac{{}^{A}\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \sum_{i=1}^{3} \frac{\mathrm{d}v_i}{\mathrm{d}t}\mathbf{b}_i + \sum_{i=1}^{3} v_i \frac{{}^{A}\mathrm{d}\mathbf{b}_i}{\mathrm{d}t} \tag{26}$$

$$= \frac{{}^{B}\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \sum_{i=1}^{3} v_i \, {}^{A}\omega^{B} \times \mathbf{b}_i \tag{27}$$

Differentiation in Two Reference Frames

$$\frac{{}^{A}\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \sum_{i=1}^{3} \frac{\mathrm{d}v_i}{\mathrm{d}t}\mathbf{b}_i + \sum_{i=1}^{3} v_i \frac{{}^{A}\mathrm{d}\mathbf{b}_i}{\mathrm{d}t} \tag{28}$$

$$= \frac{{}^{B}\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \sum_{i=1}^{3} v_i \; {}^{A}\omega^{B} \times \mathbf{b}_i \tag{29}$$

$$= \frac{{}^{B}\mathrm{d}\mathbf{v}}{\mathrm{d}t} + {}^{A}\omega^{B} \times \sum_{i=1}^{3} v_i\mathbf{b}_i \tag{30}$$

$$= \frac{{}^{B}\mathrm{d}\mathbf{v}}{\mathrm{d}t} + {}^{A}\omega^{B} \times \mathbf{v} \tag{31}$$

Differentiation in Two Reference Frames

$$\frac{{}^{A}\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{{}^{B}\mathrm{d}\mathbf{v}}{\mathrm{d}t} + {}^{A}\omega^{B} \times \mathbf{v} \tag{32}$$

The angular velocity of a rigid body B in a reference frame A can be expressed in the following form involving n auxiliary reference frames $A_1, \ldots, A_n$:

Auxiliary Reference Frames

$$ {}^{A}\omega^{B} = {}^{A}\omega^{A_1} + {}^{A_1}\omega^{A_2} + \cdots + {}^{A_{n-1}}\omega^{A_n} + {}^{A_n}\omega^{B} \tag{33} $$

The reference frames $A_1, \ldots, A_n$ may or may not correspond to actual rigid bodies. Frequently, such reference frames are introduced as aids in analysis, but have no physical counterparts.

Derivation. For any vector β fixed in B

$$ \frac{{}^{A}\mathrm{d}\beta}{\mathrm{d}t} = {}^{A}\omega^{B} \times \beta \qquad \frac{{}^{A_1}\mathrm{d}\beta}{\mathrm{d}t} = {}^{A_1}\omega^{B} \times \beta \tag{34} $$

$$ \frac{{}^{A}\mathrm{d}\beta}{\mathrm{d}t} = \frac{{}^{A_1}\mathrm{d}\beta}{\mathrm{d}t} + {}^{A}\omega^{A_1} \times \beta \tag{35} $$

Auxiliary Reference Frames

Therefore,

$$ {}^{A}\omega^{B} \times \beta = {}^{A_1}\omega^{B} \times \beta + {}^{A}\omega^{A_1} \times \beta \tag{36} $$

Since this equation is satisfied by every β fixed in B, it implies that

$$ {}^{A}\omega^{B} = {}^{A_1}\omega^{B} + {}^{A}\omega^{A_1} \tag{37} $$

Proceeding similarly, one can verify that

Auxiliary Reference Frames

$$ {}^{A}\omega^{B} = {}^{A}\omega^{A_1} + {}^{A_1}\omega^{A_2} + \cdots + {}^{A_{n-1}}\omega^{A_n} + {}^{A_n}\omega^{B} \tag{38} $$

Angular Acceleration

The angular acceleration ${}^{A}\alpha^{B}$ of a rigid body B in a reference frame A is defined as the first time-derivative in A of the angular velocity of B in A

$$ {}^{A}\alpha^{B} \triangleq \frac{{}^{A}\mathrm{d}{}^{A}\omega^{B}}{\mathrm{d}t} \tag{39} $$

We also know that

$$ \frac{{}^{A}\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{{}^{B}\mathrm{d}\mathbf{v}}{\mathrm{d}t} + {}^{A}\omega^{B} \times \mathbf{v} \tag{40} $$

$$ {}^{A}\alpha^{B} = \frac{{}^{A}\mathrm{d}{}^{A}\omega^{B}}{\mathrm{d}t} = \frac{{}^{B}\mathrm{d}{}^{A}\omega^{B}}{\mathrm{d}t} + {}^{A}\omega^{B} \times {}^{A}\omega^{B} = \frac{{}^{B}\mathrm{d}{}^{A}\omega^{B}}{\mathrm{d}t} \tag{41} $$

The angular velocity of B in A can always be expressed as ${}^{A}\omega^{B} = \omega\mathbf{k}_{\omega}$, where $\mathbf{k}_{\omega}$ is a unit vector parallel to ${}^{A}\omega^{B}$; similarly ${}^{A}\alpha^{B}$ can always be expressed as ${}^{A}\alpha^{B} = \alpha\mathbf{k}_{\alpha}$, where $\mathbf{k}_{\alpha}$ is a unit vector parallel to ${}^{A}\alpha^{B}$. In general, $\mathbf{k}_{\omega}$ differs from $\mathbf{k}_{\alpha}$.

Let $\mathbf{p}$ denote the position vector from any point O fixed in a reference frame A to a point P moving in A. The velocity of P in A and the acceleration of P in A, denoted by ${}^{A}\mathbf{v}^{P}$ and ${}^{A}\mathbf{a}^{P}$ 9 respectively, are defined as

$$ {}^{A}\mathbf{v}^{P} \triangleq \frac{{}^{A}\mathrm{d}\mathbf{p}}{\mathrm{d}t} \tag{42} $$

and

$$ {}^{A}\mathbf{a}^{P} \triangleq \frac{{}^{A}\mathrm{d}\,{}^{A}\mathbf{v}^{P}}{\mathrm{d}t} \tag{43} $$

Example

P_1 and P_2 designate two points connected by a line of length L and free to move in a plane B that is rotating at a constant rate ω about a line Y fixed both in B and in a reference frame A. The velocities ${}^A\mathbf{v}^{P_1}$ and ${}^A\mathbf{v}^{P_2}$ of P_1 and P_2 in A are to be expressed in terms of the quantities q_1, q_2, q_3 their time-derivatives $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$, and the mutually perpendicular unit vectors e_x, e_y, e_z.

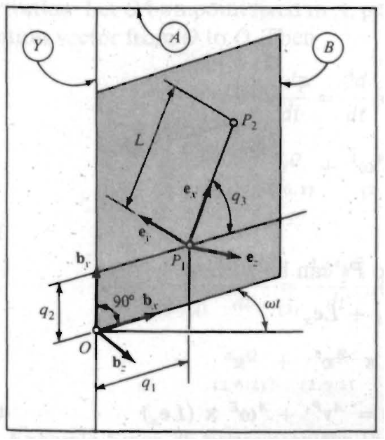


Figure: Velocity.

Velocity and Acceleration

Example

The point O is fixed in A, and the position vector $\mathbf{p}_1$ from O from P_1 is

$$\mathbf{p}_1 = q_1\mathbf{b}_x + q_2\mathbf{b}_y \qquad (44)$$

$$^{A}\mathbf{v}^{P_1} = \frac{^{A}\mathrm{d}\mathbf{p}_1}{\mathrm{d}t} = \frac{^{B}\mathrm{d}\mathbf{p}_1}{\mathrm{d}t} + {}^{A}\omega^{B} \times \mathbf{p}_1$$

where

$$\frac{^{A}\mathrm{d}\mathbf{p}_1}{\mathrm{d}t} = \dot{q}_1\mathbf{b}_x + \dot{q}_2\mathbf{b}_y$$

$$^{A}\omega^{B} \times \mathbf{p}_1 = \omega\mathbf{b}_y \times (q_1\mathbf{b}_x + q_2\mathbf{b}_y) = -\omega q_1 \mathbf{b}_z$$

Therefore,

$$^{A}\mathbf{v}^{P_1} = \dot{q}_1\mathbf{b}_x + \dot{q}_2\mathbf{b}_y - \omega q_1\mathbf{b}_z \qquad (45)$$

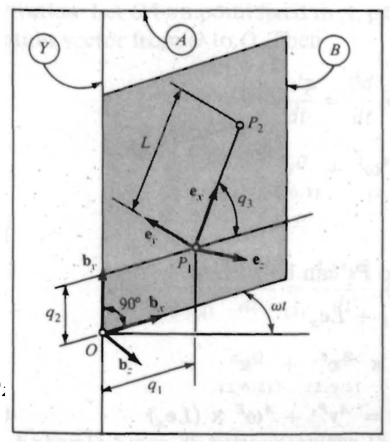


Figure: Velocity.

Example The unit vectors $\mathbf{b}_{x,y,z}$ are related to the unit vectors $\mathbf{e}_{x,y,z}$ through

	$\mathbf{e}_x$	$\mathbf{e}_y$	$\mathbf{e}_z$
$\mathbf{b}_x$	$\cos q_3$	$-\sin q_3$	0
$\mathbf{b}_y$	$\sin q_3$	$\cos q_3$	0
$\mathbf{b}_z$	0	0	1

$$^{A}\mathbf{v}^{P_1} = (\dot{q}_1 \cos q_3 + \dot{q}_2 \sin q_3)\mathbf{e}_x$$
$$+(-\dot{q}_1 \sin q_3 + \dot{q}_2 \cos q_3)\mathbf{e}_y - \omega q_1 \mathbf{e}_z$$

which is the desired expression for the velocity of P_1 in A. Proceeding similary, one can determine the velocity of P_2 in A.

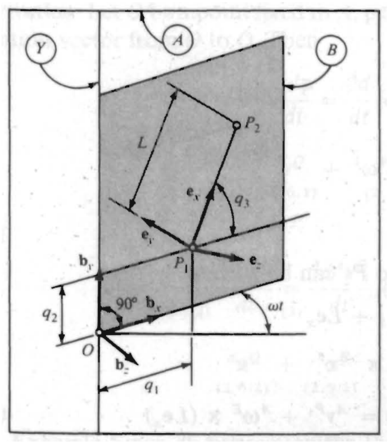


Figure: Velocity.

Thank You

Thank You!
Questions please