Robotics

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Position and Orientation of Rigid Bodies

Position: ${}^{A}\mathbf{p}^{AB} \in \mathbb{R}^{3\times 1}$ in A.
Orientation: ${}^{A}\mathbf{R}^{B} \in \mathbb{R}^{3\times 3}$ in A.

$$\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1} \;\;\Rightarrow\;\; {}^{A}\mathbf{R}^{B}\,{}^{B}\mathbf{R}^{A} = \mathbf{I} \quad |\,\mathbf{R}\,| = +1$$

$${}^{A}\mathbf{R}^{B} = \left[{}^{A}\mathbf{x}^{B} \quad {}^{A}\mathbf{y}^{B} \quad {}^{A}\mathbf{z}^{B}\right]$$

- $\mathbf{x}_A$ $\mathbf{y}_A$ $\mathbf{z}^A$ are unit vectors (with unitary norm) of frame A
- $\mathbf{x}_B$ $\mathbf{y}_B$ $\mathbf{z}^B$ are unit vectors (with unitary norm) of frame B
- vectors of ${}^{A}\mathbf{x}^{B}$ are the direction cosines of the axes of B with respect to (w.r.t.) A

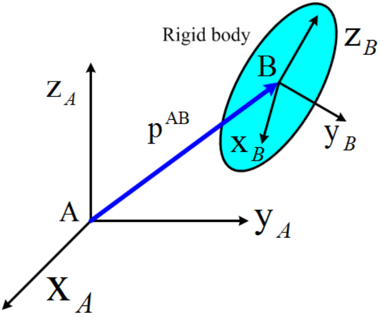


Figure: Rotation of rigid body

$$
{}^{A}\mathbf{R}^{B} = \begin{bmatrix} \mathbf{x}_A^{\mathrm{T}}\mathbf{x}_B & \mathbf{x}_A^{\mathrm{T}}\mathbf{y}_B & \mathbf{x}_A^{\mathrm{T}}\mathbf{z}_B \\ \mathbf{y}_A^{\mathrm{T}}\mathbf{x}_B & \mathbf{y}_A^{\mathrm{T}}\mathbf{y}_B & \mathbf{y}_A^{\mathrm{T}}\mathbf{z}_B \\ \mathbf{z}_A^{\mathrm{T}}\mathbf{x}_B & \mathbf{z}_A^{\mathrm{T}}\mathbf{y}_B & \mathbf{z}_A^{\mathrm{T}}\mathbf{z}_B \end{bmatrix}
$$

where $\mathbf{x}_A^{\mathrm{T}}\mathbf{z}_B$ is the direction cosine of $\mathbf{z}_B$ w.r.t. $\mathbf{x}_A$

Chain rule property

$$
{}^{k}\mathbf{R}^{i}\ {}^{i}\mathbf{R}^{j} = {}^{k}\mathbf{R}^{j}
$$

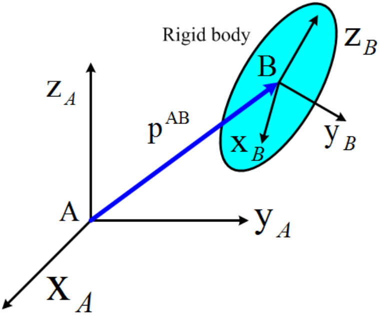


Figure: Rotation of rigid body

- ${}^{k}\mathbf{R}^{i}$ is the orientation of i w.r.t. k
- ${}^{i}\mathbf{R}^{j}$ is the orientation of j w.r.t. i
- ${}^{k}\mathbf{R}^{j}$ is the orientation of j w.r.t. k

In general, rotation matrices does not commute, ${}^{k}\mathbf{R}^{i}\ {}^{i}\mathbf{R}^{j} \neq {}^{i}\mathbf{R}^{j}\ {}^{k}\mathbf{R}^{i}$

$$
{}^{0}\mathbf{P} = \begin{bmatrix} {}^{0}\mathbf{p_x} \\ {}^{0}\mathbf{p_y} \\ {}^{0}\mathbf{p_z} \end{bmatrix} = {}^{1}\mathbf{p_x}\, {}^{0}\mathbf{x}_1 + {}^{1}\mathbf{p_y}\, {}^{0}\mathbf{y}_1 + {}^{1}\mathbf{p_z}\, {}^{0}\mathbf{z}_1
$$

$$
= \begin{bmatrix} {}^{0}\mathbf{x}_1 & {}^{0}\mathbf{y}_1 & {}^{0}\mathbf{z}_1 \end{bmatrix} \begin{bmatrix} {}^{1}\mathbf{p_x} \\ {}^{1}\mathbf{p_y} \\ {}^{1}\mathbf{p_z} \end{bmatrix}
$$

$$
= {}^{0}\mathbf{R}^{1}\, {}^{1}\mathbf{P}
$$

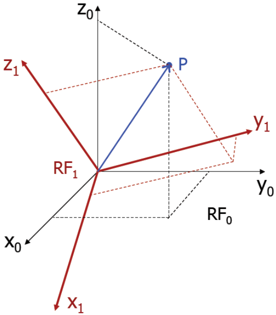


Figure: Vector in two reference frames

The rotation matrix ${}^{0}\mathbf{R}^{1}$ (i.e., the orientation of reference frame 1 w.r.t. reference frame 0) represents also the change of coordinates of a vector from frame 1 to frame 0

$$\begin{aligned} x &= OBxB = u\cos\theta - v\sin\theta \\ y &= OC + Cy = u\sin\theta + v\cos\theta \\ z &= w \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{R}_z(\theta) \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

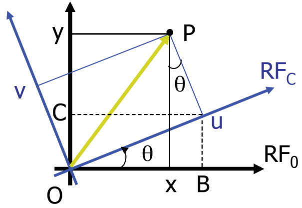


Figure: Simple rotation

$$^{0}\mathbf{OP} = \mathbf{R}_z(\theta)\ ^{\mathrm{C}}\mathbf{OP}$$

$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad \mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_z(-\theta) = \mathbf{R}_z^{\mathrm{T}}(\theta) \qquad \mathbf{R}_x(-\theta) = \mathbf{R}_x^{\mathrm{T}}(\theta) \qquad \mathbf{R}_y(-\theta) = \mathbf{R}_y^{\mathrm{T}}(\theta)$$

$$\mathrm{x} = \mid \mathrm{v} \mid \cos\alpha \qquad \mathrm{y} = \mid \mathrm{v} \mid \sin\alpha$$

$$\begin{aligned} \mathrm{x}' &= \mid \mathrm{v} \mid \cos(\theta+\alpha) \\ &= \mid \mathrm{v} \mid (\cos\alpha\cos\theta - \sin\alpha\sin\theta) \\ &= \mathrm{x}\cos\theta - \mathrm{y}\sin\theta \end{aligned}$$

$$\begin{aligned} \mathrm{y}' &= \mid \mathrm{v} \mid \sin(\theta+\alpha) \\ &= \mid \mathrm{v} \mid (\sin\alpha\cos\theta + \cos\alpha\sin\theta) \\ &= \mathrm{x}\sin\theta + \mathrm{y}\cos\theta \end{aligned}$$

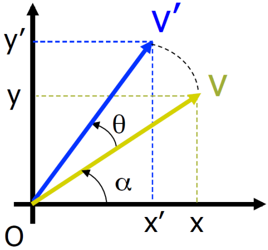
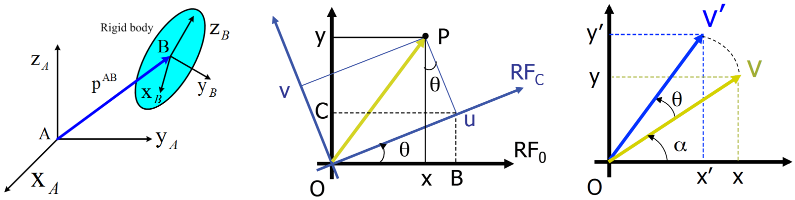


Figure: Simple rotation

$$\mathrm{z}' = \mathrm{z}$$

$$\mathrm{v}' = \begin{bmatrix} \mathrm{x}' \\ \mathrm{y}' \\ \mathrm{z}' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix} = \mathbf{R}_z(\theta) \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix} = \mathbf{R}_z(\theta)\mathrm{v}$$

The same rotation matrix, e.g., $\mathbf{R}_z(\theta)$, can be represented as follows:



(a) the orientation of a rigid body with respect to a reference frame 0

(b) the change of coordinates from C to 0

(c) the vector rotation operator

Figure: The rotation matrix ${}^{0}\mathbf{R}^{\mathrm{C}}$ is an operator superposing frame 0 to frame C.

Composition of rotations

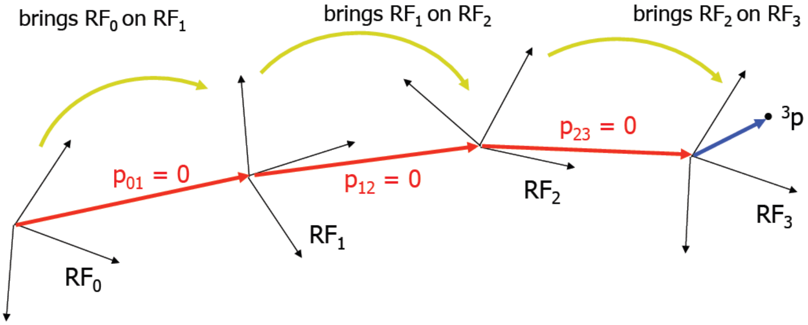


Figure: Composition of rotations

$$ {}^0\mathbf{p} = \left({}^0\mathbf{R}^1\ {}^1\mathbf{R}^2\ {}^2\mathbf{R}^3\right)\ {}^3\mathbf{p} = {}^0\mathbf{R}^3\ {}^3\mathbf{p} $$ 63 products and 42 summations

$$ {}^0\mathbf{p} = {}^0\mathbf{R}^1 \left({}^1\mathbf{R}^2 \left({}^2\mathbf{R}^3\ {}^3\mathbf{p}\right)\right) $$ 27 products and 18 summations

Axis/Angle Representation of Rotations

Rotation $\mathbf{R}(\mathbf{r}, \theta)$ between two reference frames can be represented a sequence of three rotations:

$$\mathbf{R}(\mathbf{r}, \theta) = \mathbf{C}\mathbf{R}_{\mathrm{z}}(\theta)\mathbf{C}^{\mathrm{T}}$$

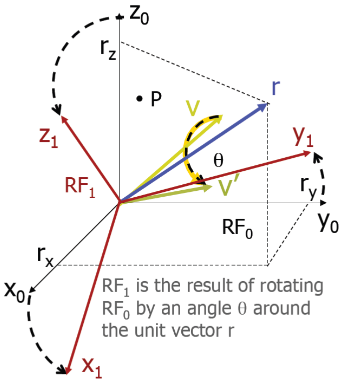


Figure: Representation of Rotations

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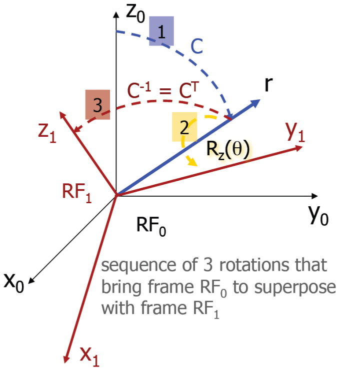


Figure: Representation of Rotations

Based on a unit vector $\mathbf{r}$ ($\|\mathbf{r}\| = 1$) and θ, we seek the following:

$$\mathbf{R}(\mathbf{r}, \theta) = \left[\begin{array}{ccc} {}^0\mathbf{x}^1 & {}^0\mathbf{y}^1 & {}^0\mathbf{z}^1 \end{array} \right]$$

such that

$${}^0\mathbf{P} = \mathbf{R}(\mathbf{r}, \theta)\ {}^1\mathbf{P} \qquad {}^0\mathbf{V}' = \mathbf{R}(\mathbf{r}, \theta)\ {}^0\mathbf{V}$$

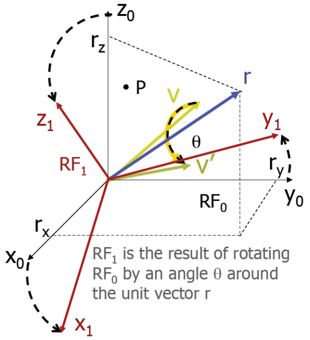


Figure: Sequence of three rotations

$$\mathbf{R}(\mathbf{r}, \theta) = \mathbf{C}\mathbf{R}_z(\theta)\mathbf{C}^T$$

$\mathbf{R}(\mathbf{r}, \theta)$ is a sequence of three rotations, z-axis coincides with $\mathbf{r}$ after the first rotation

$$\mathbf{C} = [\mathbf{n} \quad \mathbf{s} \quad \mathbf{r}]$$

$\mathbf{n}$ and $\mathbf{s}$ are orthogonal unit vectors such that

$$\begin{aligned} \mathbf{n} \times \mathbf{s} &= \mathbf{r} \\ n_y s_z - s_y n_z &= r_x \\ n_z s_x - s_z n_x &= r_y \\ n_x s_y - s_x n_y &= r_z \end{aligned}$$

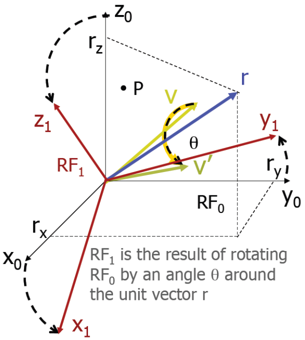


Figure: Sequence of three rotations

Position and Orientation of Rigid Bodies

$$\mathbf{R}(\mathbf{r},\theta) = \begin{bmatrix}\mathbf{n} & \mathbf{s} & \mathbf{r}\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{n}^{\mathrm{T}}\\ \mathbf{s}^{\mathrm{T}}\\ \mathbf{r}^{\mathrm{T}}\end{bmatrix}$$

$$\mathbf{R}(\mathbf{r},\theta) = \mathbf{r}\mathbf{r}^{\mathrm{T}} + \left(\mathbf{n}\mathbf{n}^{\mathrm{T}} + \mathbf{s}\mathbf{s}^{\mathrm{T}}\right)\cos\theta + \left(\mathbf{s}\mathbf{n}^{\mathrm{T}} - \mathbf{n}\mathbf{s}^{\mathrm{T}}\right)\sin\theta$$

Taking the following into account:

$$\mathbf{C}\mathbf{C}^{\mathrm{T}} = \mathbf{n}\mathbf{n}^{\mathrm{T}} + \mathbf{s}\mathbf{s}^{\mathrm{T}} + \mathbf{r}\mathbf{r}^{\mathrm{T}} = \mathbf{I}$$

$$\mathbf{s}\mathbf{n}^{\mathrm{T}} - \mathbf{n}\mathbf{s}^{\mathrm{T}} = \begin{bmatrix}0 & -r_z & r_y\\ r_z & 0 & -r_x\\ -r_y & r_x & 0\end{bmatrix} = \mathbf{S}(\mathbf{r})$$

We obtain

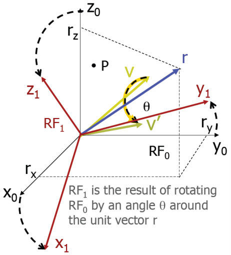


Figure: Sequence of three rotations

Rodriguez Formula

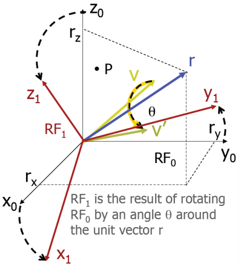
$$\mathbf{R}(\mathbf{r},\theta) = \mathbf{r}\mathbf{r}^{\mathrm{T}} + \left(\mathbf{I} - \mathbf{r}\mathbf{r}^{\mathrm{T}}\right)\cos\theta + \mathbf{S}(\mathbf{r})\sin\theta$$

Rodriguez Formula

$$\mathbf{R}(\mathbf{r},\theta) = \mathbf{r}\mathbf{r}^{\mathrm{T}} + \left(\mathbf{I} - \mathbf{r}\mathbf{r}^{\mathrm{T}}\right)\cos\theta + \mathbf{S}(\mathbf{r})\sin\theta$$

- $\mathbf{R}(\mathbf{r},\theta)$ depends only on $\mathbf{r}$ and θ
- $\mathbf{R}^{\mathrm{T}}(\mathbf{r},-\theta) = \mathbf{R}(-\mathbf{r},-\theta)$

$$\mathbf{R} = \begin{bmatrix} r_x^2(1-c\theta)+c\theta & r_xr_y(1-c\theta)-r_zs\theta & r_xr_z(1-c\theta)+r_ys\theta \\ r_xr_y(1-c\theta)+r_zs\theta & r_y^2(1-c\theta)+c\theta & r_yr_z(1-c\theta)-r_xs\theta \\ r_xr_z(1-c\theta)-r_ys\theta & r_yr_z(1-c\theta)+r_xs\theta & r_z^2(1-c\theta)+c\theta \end{bmatrix}$$



$$\mathbf{R}(\mathbf{r},\theta) = \mathbf{r}\mathbf{r}^{\mathrm{T}} + \left(\mathbf{I} - \mathbf{r}\mathbf{r}^{\mathrm{T}}\right)\cos\theta + \mathbf{S}(\mathbf{r})\sin\theta$$

$$\mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{z}_0$$

$$\begin{aligned} \mathbf{R}(\mathbf{r},\theta) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cos\theta + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\sin\theta \\ &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{R}_z(\theta) \end{aligned}$$

Properties of $\mathbf{R}(\mathbf{r}, \theta)$

- $\mid \mathbf{R}(\mathbf{r}, \theta) \mid = +1$
- $\text{tr}(\mathbf{R}(\mathbf{r}, \theta)) = \text{tr}\left(\mathbf{r}\mathbf{r}^{\mathrm{T}}\right) + \text{tr}\left(\mathbf{I} - \mathbf{r}\mathbf{r}^{\mathrm{T}}\right)\cos\theta = 1 + 2\cos\theta$
- $\mathbf{R}(\mathbf{r}, \theta)\mathbf{r} = \mathbf{r}$ ($\mathbf{r}$ is the invariant axis in this rotation)
- when $\mathbf{r}$ is one of the coordinate axes, $\mathbf{R}(\mathbf{r}, \theta)$ is nothing but one of the known elementary rotation matrices

Inverse Problem

Given a rotation matrix $\mathbf{R}$, Find a unit vector $\mathbf{r}$ and an angle θ such that

$$\mathbf{R}(\mathbf{r},\theta) = \mathbf{r}\mathbf{r}^{\mathrm{T}} + \left(\mathbf{I} - \mathbf{r}\mathbf{r}^{\mathrm{T}}\right)\cos\theta + \mathbf{S}(\mathbf{r})\sin\theta$$

Since $tr(\mathbf{R}) = R_{11} + R_{22} + R_{33} = 1 + 2\cos\theta$. Therefore,

$$\theta = \arccos\left(\frac{R_{11} + R_{22} + R_{33} - 1}{2}\right)$$

- provides only values in $[0,\pi]$ (thus, never negative angles θ)
- loss of numerical accuracy for $\theta \to 0$

Inverse Problem

Let us try the following:

$$\mathbf{R}-\mathbf{R}^{\mathrm{T}} = \begin{bmatrix} 0 & R_{12}-R_{21} & R_{13}-R_{31} \\ R_{21}-R_{12} & 0 & R_{23}-R_{32} \\ R_{31}-R_{13} & R_{32}-R_{23} & 0 \end{bmatrix} = 2\sin\theta \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

$$\sin\theta = \pm\frac{1}{2}\sqrt{(R_{12}-R_{21})^2+(R_{13}-R_{31})^2+(R_{23}-R_{32})^2}$$

Inverse Problem

$$\theta = \mathrm{atan2}\left(\pm\sqrt{(R_{12}-R_{21})^2+(R_{13}-R_{31})^2+(R_{23}-R_{32})^2}, R_{11}+R_{22}+R_{33}-1\right)$$

$$\mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \frac{1}{2\sin\theta}\begin{bmatrix} R_{32}-R_{23} \\ R_{13}-R_{31} \\ R_{21}-R_{12} \end{bmatrix}$$

atan2 function

- atan2 with output values in four quadrants, takes two input arguments, takes values in $[-\pi, \pi]$, and is undefined only for $(0, 0)$
- uses the sign of both arguments to define the output quadrant
- based on arctan function with output values in $[-\pi/2, +\pi/2]$

$$\mathrm{atan2}(y, x) = \begin{cases} \arctan(\frac{y}{x}), & x > 0; \\ \pi + \arctan(\frac{y}{x}), & y \geq 0, x < 0; \\ -\pi + \arctan(\frac{y}{x}), & y < 0, x < 0; \\ \frac{\pi}{2}, & y > 0, x = 0; \\ -\frac{\pi}{2}, & y < 0, x = 0; \\ \text{undefined}, & y = 0, x = 0. \end{cases}$$

Task Space and Joint Space

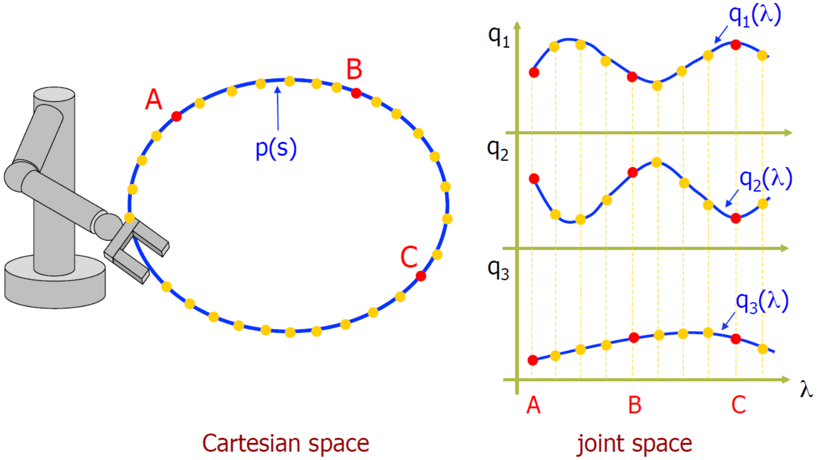


Figure: Problem formulation

Thank You!
Questions please