Sliding mode control

1) New title: Sliding mode control

2) Nonlinear control strategies based on sliding mode are used to deal with parameter and model uncertainties.

3) In sliding mode control, a Lyapunov approach is used to keep the nonlinear system under control.

4) In sliding mode control, a higher order system is transformed into a lower order system.

5) Reasons for modeling uncertainties:

- Actual uncertainty about the plant (unknown parameters)
- Simplified model design (purposeful simplification)

6) Types of modeling uncertainties:

- Structured (parametric)
- Unstructured (unmodeled dynamics, inaccuracy on the system order)

7) Robust control: An approach to deal with modeling uncertainties.

8) A robust controller is usually composed of a nominal part (a feedback controller) and a corrective term (dealing with model uncertainty.)

9) Sliding mode control is a robust control strategy.

10) In sliding mode control a switching control law is used.

11) The switching law is used to drive the state trajectory onto a prespecified surface.

12) Terminology: This surface is called a switching surface, sliding surface or sliding manifold.

13) In many cases this surface is a line (for second order SISO systems).

14) Ideally, once intercepted, the sliding surface becomes positively invariant for system states. (System states slide along the sliding surface.)

15) Important task: Design of a switching control that will drive state trajectories to the sliding surface, and keep them on the sliding surface.

16) This is done by a Lyapunov approach.

17) In sliding mode control, the Lyapunov function is defined in terms of the sliding surface.

18) Consider the following single input nonlinear system:

$$x^{(n)} = f(X,t) + b(X,t)u(t) \qquad (1)$$

$X(t)$: State vector

$u(t)$: Control input

x : Output state of interest. (The other states in X are higher order derivatives of x up to the order $(n-1)$.)

$f(X,t)$, $b(X,t)$: Nonlinear functions.

19) $f(X,t)$ is not exactly known. The imprecision of our knowledge on $f(X,t)$ is upper bounded by a known continuous function of X.

20) $b(X,t)$ is not exactly known. $b(X,t)$ is of known sign. $b(X,t)$ is upper and lower bounded by known functions of X.

21) Control problem: We want that X tracks the time-varying state reference X_d in the presence of model imprecision on $f(X,t)$ and $b(X,t)$.

22) Define the sliding variable $s(t)$ as:

$$s(t) = (\frac{d}{dt} + \lambda)^{n-1} \tilde{x}(t) \tag{2}$$

λ : Strictly positive constant

$\tilde{x}(t) = x(t) - x_d(t)$

$x_d(t)$: Desired output state

23) The sliding surface S is defined by equating the sliding variable to 0 :

$S \equiv \{\tilde{X} : s(t) = 0\}$

24) The system behavior on the sliding surface is called sliding mode or sliding regime.

25) Remark: The tracking problem for the n-dimensional state vector is replaced by a first-order stabilization problem for $s(t)$.

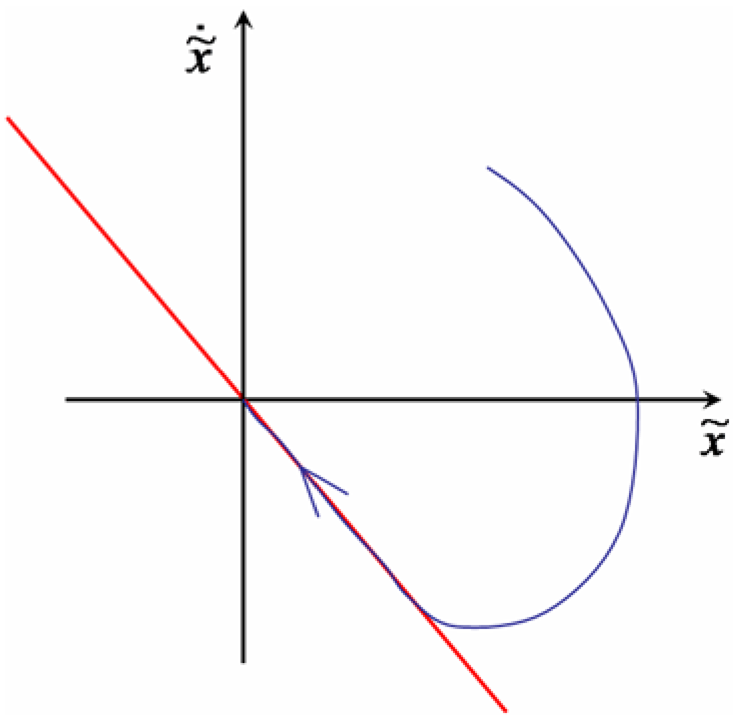
26) Since λ is positive, the error dynamics on the sliding line is Hurwitz and the tracking error decays to zero with a speed dictated by λ.

$$0 = s(t) = (\frac{d}{dt} + \lambda)^{n-1} \tilde{x}(t)$$

27) Moving s to zero can be achieved if the control u is designed in such a way that the following inequality holds:

$$\tfrac{1}{2} \frac{d}{dt} s^2 \leq -\eta |s| \tag{4}$$

η: a positive constant.



28) Question: With $\frac{1}{2}\frac{d}{dt}s^2 \le -\eta|s|$, if the initial states are off the sliding surface, how fast do they converge to the sliding surface?

29) Answer:
Suppose that the system state is not on the sliding surface.
Then $s \ne 0$.
Suppose that $s > 0$.

$$\frac{1}{2}\frac{d}{dt}s^2 \le -\eta|s|. \Rightarrow s\dot{s} \le -\eta|s|.$$

$$s > 0. \Rightarrow s = |s|. \Rightarrow \dot{s} \le -\eta.$$

Now integrate both sides of this equation:

$$\int_0^{t_{reach}} \dot{s}dt \le \int_0^{t_{reach}} -\eta dt$$

t_{reach}: State reaching time to the sliding surface.

$$\Rightarrow s\Big|_0^{t_{reach}} \le -\eta t\Big|_0^{t_{reach}}. \Rightarrow s(t_{reach}) - s(0) = 0 - s(0) \le -\eta(t_{reach} - 0) \quad \Rightarrow t_{reach} \le \frac{s(0)}{\eta}$$

If we start by the assumption that $s < 0$. Then, a similar result will be obtained for the reaching time:

$$t_{reach} \le \left|\frac{s(0)}{\eta}\right|.$$

30) Remark: Starting from any initial state error, the state error trajectories reach the sliding surface in a finite time less than $\left|\frac{s(0)}{\eta}\right|$ and then slide along the surface to the origin exponentially with a time constant $\frac{1}{\lambda}$.

31) Question: How should the control u be designed in order to achieve

$$\tfrac{1}{2}\frac{d}{dt}s^2 \leq -\eta|s| ?$$

32) Consider the following second order system:

$$\ddot{x}(t) = f(X,t) + u(t) \tag{5}$$

33) Let $\hat{f}(X,t)$ be an estimate for $f(X,t)$ with

$$\left|\hat{f}(X,t) - f(X,t)\right| \leq F(X,t) . \tag{6}$$

34) Define the sliding variable as

$$s(t) = \left(\frac{d}{dt} + \lambda\right)\tilde{x} = \dot{\tilde{x}} + \lambda\tilde{x} .$$

35) Differentiate the sliding variable:

$$\dot{s}(t) = \ddot{x} - \ddot{x}_d + \lambda\dot{\tilde{x}} . \tag{8}$$

$$\Rightarrow \dot{s}(t) = f(X,t) + u(t) - \ddot{x}_d + \lambda\dot{\tilde{x}} . \tag{9}$$

36) Define the approximation control law

$$\hat{u}(t) = -\hat{f}(X,t) + \ddot{x}_d - \lambda\dot{\tilde{x}}\ . \qquad (10)$$

37) $\hat{u}(t)$ is designed to achieve $\dot{s}(t) = 0$.
$\hat{u}(t)$ is the best estimate of "equivalent control".
Equivalent control: Control for keeping s at a constant value.
This constant value is 0 on the sliding line.

38) Remember the target reaching regime:

$$\tfrac{1}{2}\frac{d}{dt}\left(s^2(t)\right) \le -\eta\left|s(t)\right| \qquad \eta > 0 \qquad (11)$$

39) In order to achieve this regime with the existing uncertainty on $f(X,t)$, the following control law is designed.

$$u(t) = \hat{u}(t) - k(X,t)\operatorname{sgn}(s(t)) \qquad (12)$$

$$k(X,t) = F(X,t) + \eta$$

40) How does this design for u achieve $\frac{1}{2}\frac{d}{dt}\left(s^2(t)\right) \le -\eta\left|s(t)\right|$?

41) Answer:

$$
\begin{aligned}
\tfrac{1}{2}\frac{d}{dt}(s^2(t)) &= \dot{s}(t)s(t) \\
&= (f(X,t) + u(t) - \ddot{x}_d + \lambda\dot{\tilde{x}})s(t) \\
&= (f(X,t) + \hat{u}(t) - k(X,t)\,\mathrm{sgn}(s(t)) - \ddot{x}_d + \lambda\dot{\tilde{x}})s(t) \\
&= (f(X,t) - \hat{f}(X,t) + \ddot{x}_d - \lambda\dot{\tilde{x}} - k(X,t)\,\mathrm{sgn}(s(t)) - \ddot{x}_d + \lambda\dot{\tilde{x}})s(t) \qquad (13) \\
&= (f(X,t) - \hat{f}(X,t) - k(X,t)\,\mathrm{sgn}(s(t)))s(t) \\
&= (f(X,t) - \hat{f}(X,t))s(t) - k(X,t)\left|s(t)\right| \\
&= (f(X,t) - \hat{f}(X,t))s(t) - (F(X,t) + \eta)\left|s(t)\right| \le -\eta\left|s(t)\right|
\end{aligned}
$$

42) Consider the following more general second order system:

$$\ddot{x}(t) = f(X,t) + b(X,t)u(t)$$

$b(X,t)$ is bounded as: $0 \le b_{min}(X,t) \le b(X,t) \le b_{max}(X,t)$

43) In the control design we can use the geometric mean of the lower and upper bounds can be used as the estimate for $b(X,t)$:

$$\hat{b}(X,t) = \sqrt{b_{min}(X,t)\, b_{max}(X,t)}$$

44) Let $\beta \equiv \sqrt{\frac{b_{max}}{b_{min}}}$. Then the bound can be expressed as $\beta^{-1} \le \frac{\hat{b}}{b} \le \beta$.

45) Question: How is that so? Assume bmin = b then bmax = b

46) Answer: $\frac{\hat{b}}{b} = \frac{\sqrt{b_{min}\, b_{max}}}{b} = \frac{\sqrt{b_{min}}\sqrt{b_{max}}}{b} = \frac{\sqrt{b_{min}}\sqrt{b_{max}}}{\sqrt{b}\sqrt{b}} \le \frac{\sqrt{b_{max}}}{\sqrt{b}} \le \frac{\sqrt{b_{max}}}{\sqrt{b_{min}}} = \beta$.

Similarly: $\frac{\hat{b}}{b} = \frac{\sqrt{b_{min}\, b_{max}}}{b} = \frac{\sqrt{b_{min}}\sqrt{b_{max}}}{b} = \frac{\sqrt{b_{min}}\sqrt{b_{max}}}{\sqrt{b}\sqrt{b}} \ge \frac{\sqrt{b_{min}}}{\sqrt{b}} \ge \frac{\sqrt{b_{min}}}{\sqrt{b_{max}}} = \beta^{-1}$.

47) Remark: The control law

$$u(t) = \left(\hat{b}(X,t)\right)^{-1}\left[\hat{u}(t) - k(X,t)\,\mathrm{sgn}(s(t))\right] \tag{15}$$

with

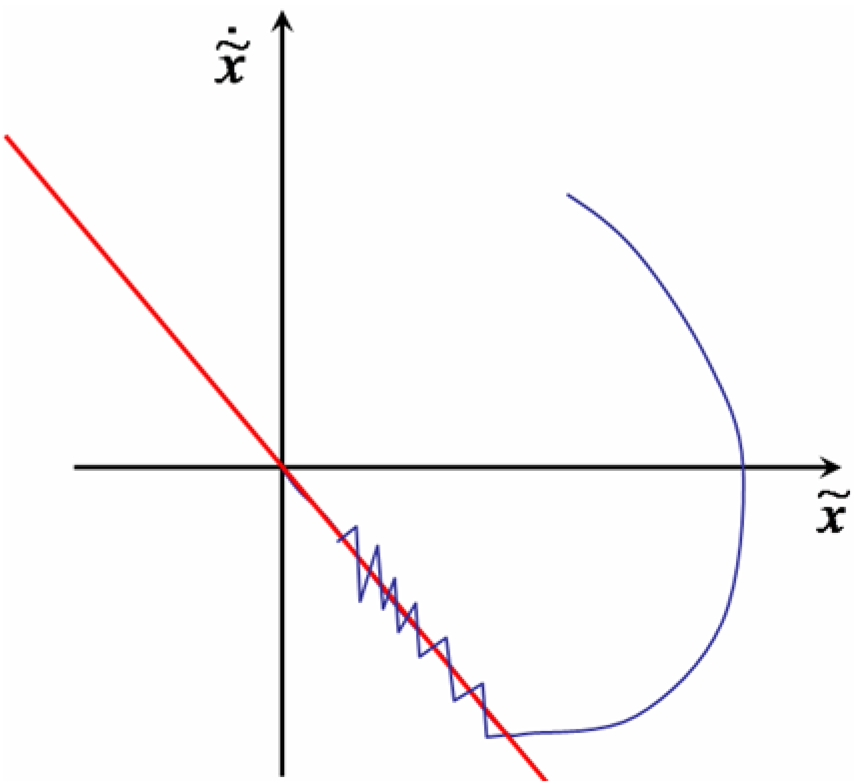
$$k(X,t) \geq \beta(X,t)(F(X,t)+\eta) + (\beta(X,t)-1)\left|\hat{u}(t)\right| \tag{16}$$

satisfies the sliding condition

$$\tfrac{1}{2}\frac{d}{dt}\left(s^2(t)\right) \leq -\eta\left|s(t)\right| .$$

48) Remark: Ideal sliding mode requires infinite frequency switching. This is not possible. Therefore zigzag behavior is observed about the sliding line.

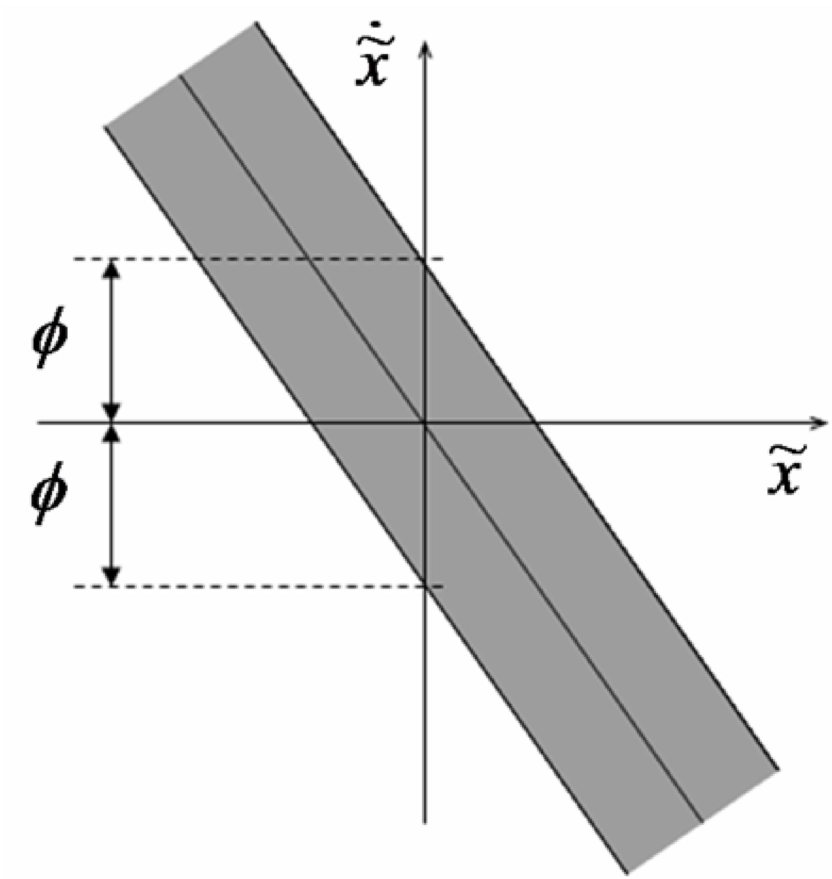
49) Definition: This behavior is called chattering.



50) Remark: In many cases chattering has to be eliminated for proper plant operation.

51) Chattering can be eliminated by smoothing the control in a narrow boundary layer around the sliding surface.

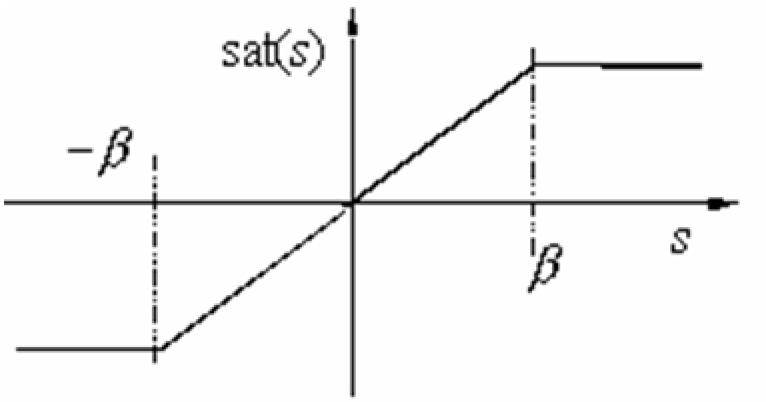
$$B = \{x : |s(\tilde{x}, t)| \leq \phi\} \qquad \phi > 0 \tag{17}$$



52) Terminology: ϕ is called the boundary layer thickness.

53) Definition: For a second order plant, the width of the boundary layer is as $\varepsilon = \dfrac{\phi}{\lambda}$.

54) Smoothing inside the boundary layer can be achieved by using the sat function in place of the sgn function in the control law.



55) Remark: The sat function is equivalent to $\dfrac{s}{\phi}$ inside the boundary layer.

56) Perfect tracking cannot be guaranteed but steady state tracking error less than the boundary layer width can be achieved.