Sliding mode control
1) New title: Sliding mode control

2) Nonlinear control strategies based on sliding mode are used to deal with parameter and model uncertainties.

3) In sliding mode control, a Lyapunov approach is used to keep the nonlinear system under control.

4) In sliding mode control, a higher order system is transformed into a lower order system.

5) Reasons for modeling uncertainties:
   - Actual uncertainty about the plant (unknown parameters)
   - Simplified model design (purposeful simplification)

6) Types of modeling uncertainties:
   - Structured (parametric)
   - Unstructured (unmodeled dynamics, inaccuracy on the system order)
7) Robust control: An approach to deal with modeling uncertainties.

8) A robust controller is usually composed of a nominal part (a feedback controller) and a corrective term (dealing with model uncertainty.)

9) Sliding mode control is a robust control strategy.

10) In sliding mode control a switching control law is used.

11) The switching law is used to drive the state trajectory onto a prespecified surface.

12) Terminology: This surface is called a switching surface, sliding surface or sliding manifold.

13) In many cases this surface is a line (for second order SISO systems).

14) Ideally, once intercepted, the sliding surface becomes positively invariant for system states. (System states slide along the sliding surface.)
15) Important task: Design of a switching control that will drive state trajectories to the sliding surface, and keep them on the sliding surface.

16) This is done by a Lyapunov approach.

17) In sliding mode control, the Lyapunov function is defined in terms of the sliding surface.
18) Consider the following single input nonlinear system:
\[
x^{(n)} = f(X,t) + b(X,t)u(t)
\]
(1)

\(X(t)\): State vector
\(u(t)\): Control input
\(x\): Output state of interest. (The other states in \(X\) are higher order derivatives of \(x\) up to the order \((n-1)\).)
\(f(X,t)\), \(b(X,t)\): Nonlinear functions.

19) \(f(X,t)\) is not exactly known. The imprecision of our knowledge on \(f(X,t)\) is upper bounded by a known continuous function of \(X\).

20) \(b(X,t)\) is not exactly known. \(b(X,t)\) is of known sign. \(b(X,t)\) is upper and lower bounded by known functions of \(X\).

21) Control problem: We want that \(X\) tracks the time-varying state reference \(X_d\) in the presence of model imprecision on \(f(X,t)\) and \(b(X,t)\).
22) Define the sliding variable \( s(t) \) as:

\[
s(t) = \left( \frac{d}{dt} + \lambda \right)^{n-1} \tilde{x}(t)
\]

\( \lambda \): Strictly positive constant
\( \tilde{x}(t) = x(t) - x_d(t) \)
\( x_d(t) \): Desired output state

23) The sliding surface \( S \) is defined by equating the sliding variable to 0:

\[
S = \{ \tilde{X} : s(t) = 0 \}
\]

24) The system behavior on the sliding surface is called sliding mode or sliding regime.

25) Remark: The tracking problem for the \( n \)-dimensional state vector is replaced by a first-order stabilization problem for \( s(t) \).
26) Since \( \lambda \) is positive, the error dynamics on the sliding line is Hurwitz and the tracking error decays to zero with a speed dictated by \( \lambda \).

\[
0 = s(t) = \left( \frac{d}{dt} + \lambda \right)^{n-1} \tilde{x}(t)
\]

27) Moving \( s \) to zero can be achieved if the control \( u \) is designed in such a way that the following inequality holds:

\[
\frac{1}{2} \frac{d}{dt} s^2 \leq -\eta |s|
\]  \hspace{1cm} (4)

\( \eta \): a positive constant.
28) Question: With $\frac{1}{2} \frac{d}{dt} s^2 \leq -\eta |s|$, if the initial states are off the sliding surface, how fast do they converge to the sliding surface?

29) Answer:
Suppose that the system state is not on the sliding surface. Then $s \neq 0$.
Suppose that $s > 0$.

$\frac{1}{2} \frac{d}{dt} s^2 \leq -\eta |s| \Rightarrow s \dot{s} \leq -\eta |s|.$

$s > 0 \Rightarrow s = |s| \Rightarrow \dot{s} \leq -\eta.$

Now integrate both sides of this equation:

$\int_{t_0}^{t_{reach}} \dot{s} dt \leq \int_{t_0}^{t_{reach}} -\eta dt$

$t_{reach}$: State reaching time to the sliding surface.

$\Rightarrow s\big|_{t_{reach}} - s\big|_{t_0} \leq -\eta (t_{reach} - t_0)$

$s(t_{reach}) - s(0) = 0 - s(0) \leq -\eta (t_{reach} - 0) \Rightarrow t_{reach} \leq \frac{s(0)}{\eta}$

If we start by the assumption that $s < 0$. Then, a similar result will be obtained for the reaching time:

$t_{reach} \leq \left| \frac{s(0)}{\eta} \right|$. 
30) Remark: Starting from any initial state error, the state error trajectories reach the sliding surface in a finite time less than $\frac{|s(0)|}{\eta}$ and then slide along the surface to the origin exponentially with a time constant $\frac{1}{\lambda}$. 
31) Question: How should the control $u$ be designed in order to achieve
\[ \frac{1}{2} \frac{d}{dt} s^2 \leq -\eta |s| ? \]

32) Consider the following second order system:
\[ \ddot{x}(t) = f(X,t) + u(t) \]  
(5)

33) Let $\hat{f}(X,t)$ be an estimate for $f(X,t)$ with
\[ |\hat{f}(X,t) - f(X,t)| \leq F(X,t). \]  
(6)

34) Define the sliding variable as
\[ s(t) = \left( \frac{d}{dt} + \lambda \right) \tilde{x} = \dot{x} + \lambda \ddot{x}. \]

35) Differentiate the sliding variable:
\[ \dot{s}(t) = \ddot{x} - \ddot{x}_d + \lambda \dddot{x}. \]  
(8)
\[ \Rightarrow \dot{s}(t) = f(X,t) + u(t) - \ddot{x}_d + \lambda \dddot{x}. \]  
(9)
36) Define the approximation control law
\[ \hat{u}(t) = -\hat{f}(X,t) + \dot{x}_d - \lambda \ddot{x}. \]  
\[ \text{(10)} \]

37) \( \hat{u}(t) \) is designed to achieve \( \dot{s}(t) = 0 \).
\( \hat{u}(t) \) is the best estimate of “equivalent control”.
Equivalent control: Control for keeping \( s \) at a constant value.
This constant value is 0 on the sliding line.

38) Remember the target reaching regime:
\[ \frac{1}{2} \frac{d}{dt} \left( s^2(t) \right) \leq -\eta |s(t)| \quad \eta > 0 \]
\[ \text{(11)} \]

39) In order to achieve this regime with the existing uncertainty on \( f(X,t) \), the following control law is designed.
\[ u(t) = \hat{u}(t) - k(X,t) \text{sgn}(s(t)) \]
\[ k(X,t) = F(X,t) + \eta \]
\[ \text{(12)} \]
40) How does this design for \( u \) achieve \( \frac{1}{2} \frac{d}{dt} (s^2(t)) \leq -\eta |s(t)| \)?

41) Answer:

\[
\frac{1}{2} \frac{d}{dt} (s^2(t)) = \dot{s}(t)s(t)
\]

\[
= (f(X,t) + u(t) - \dot{x}_d + \lambda \ddot{x})s(t)
\]

\[
= (f(X,t) + \hat{u}(t) - k(X,t) \text{sgn}(s(t)) - \dot{x}_d + \lambda \ddot{x})s(t)
\]

\[
= (f(X,t) - \hat{f}(X,t) + \dot{x}_d - \lambda \ddot{x} - k(X,t) \text{sgn}(s(t)) - \dot{x}_d + \lambda \ddot{x})s(t)
\]

\[
= (f(X,t) - \hat{f}(X,t) - k(X,t) \text{sgn}(s(t)))s(t)
\]

\[
= (f(X,t) - \hat{f}(X,t))s(t) - k(X,t) |s(t)|
\]

\[
= (f(X,t) - \hat{f}(X,t))s(t) - (F(X,t) + \eta) |s(t)| \leq -\eta |s(t)|
\]
42) Consider the following more general second order system:
\[ \ddot{x}(t) = f(X, t) + b(X, t)u(t) \]

\( b(X, t) \) is bounded as:
\[ 0 \leq b_{\min}(X, t) \leq b(X, t) \leq b_{\max}(X, t) \]

43) In the control design we can use the geometric mean of the lower and upper bounds can be used as the estimate for \( b(X, t) \):
\[ \hat{b}(X, t) = \sqrt{b_{\min}(X, t) b_{\max}(X, t)} \]

44) Let \( \beta = \sqrt{\frac{b_{\max}}{b_{\min}}} \). Then the bound can be expressed as \( \beta^{-1} \leq \frac{\hat{b}}{b} \leq \beta \).

45) Question: How is that so?  Assume \( b_{\min} = b \) then \( b_{\max} = b \)

46) Answer: \( \frac{\hat{b}}{b} = \frac{\sqrt{b_{\min} b_{\max}}}{b} = \frac{\sqrt{b_{\min}} \sqrt{b_{\max}}}{b} = \frac{\sqrt{b_{\min}} \sqrt{b_{\max}}}{\sqrt{b} \sqrt{b}} \leq \frac{b_{\max}}{\sqrt{b}} \leq \frac{b_{\max}}{\sqrt{b_{\min}}} = \beta \).

Similarly: \( \frac{\hat{b}}{b} = \frac{\sqrt{b_{\min} b_{\max}}}{b} = \frac{\sqrt{b_{\min}} \sqrt{b_{\max}}}{b} \geq \frac{\sqrt{b_{\min}} \sqrt{b_{\max}}}{\sqrt{b} \sqrt{b}} \geq \frac{\sqrt{b_{\min}}}{\sqrt{b_{\max}}} = \beta^{-1} \).
47) Remark: The control law

\[ u(t) = \left( \hat{b}(X, t) \right)^{-1} \left[ \hat{u}(t) - k(X, t) \, \text{sgn}(s(t)) \right] \]  

(15)

with

\[ k(X, t) \geq \beta(X, t)(F(X, t) + \eta) + (\beta(X, t) - 1) |\hat{u}(t)| \]  

(16)

satisfies the sliding condition

\[ \frac{1}{2} \frac{d}{dt} \left( s^2(t) \right) \leq -\eta |s(t)| \] .
48) Remark: Ideal sliding mode requires infinite frequency switching. This is not possible. Therefore zigzag behavior is observed about the sliding line.

49) Definition: This behavior is called chattering.
50) Remark: In many cases chattering has to be eliminated for proper plant operation.

51) Chattering can be eliminated by smoothing the control in a narrow boundary layer around the sliding surface.

\[ B = \{ x : |s(\tilde{x}, t)| \leq \phi \} \quad \phi > 0 \]  \hspace{1cm} (17)
52) Terminology: $\phi$ is called the boundary layer thickness.

53) Definition: For a second order plant, the width of the boundary layer is as $\varepsilon = \frac{\phi}{\lambda}$.

54) Smoothing inside the boundary layer can be achieved by using the sat function in place of the sgn function in the control law.

55) Remark: The sat function is equivalent to $\frac{S}{\phi}$ inside the boundary layer.

56) Perfect tracking cannot be guaranteed but steady state tracking error less than the boundary layer width can be achieved.