Tracking problem definition

- **Dynamic model of mobile robot**

$$\ddot{x} = \frac{\lambda}{m}\sin\phi + b_1 u_1 \cos\phi$$
$$\ddot{y} = -\frac{\lambda}{m}\cos\phi + b_1 u_1 \sin\phi$$
$$\ddot{\phi} = b_2 u_2$$

$$\dot{x}\sin\phi - \dot{y}\cos\phi = 0$$

$$b_1 = 1/(rm),\ b_2 = l/(rI)$$

$$u_1 = T_1 + T_2$$
$$u_2 = T_1 - T_2$$

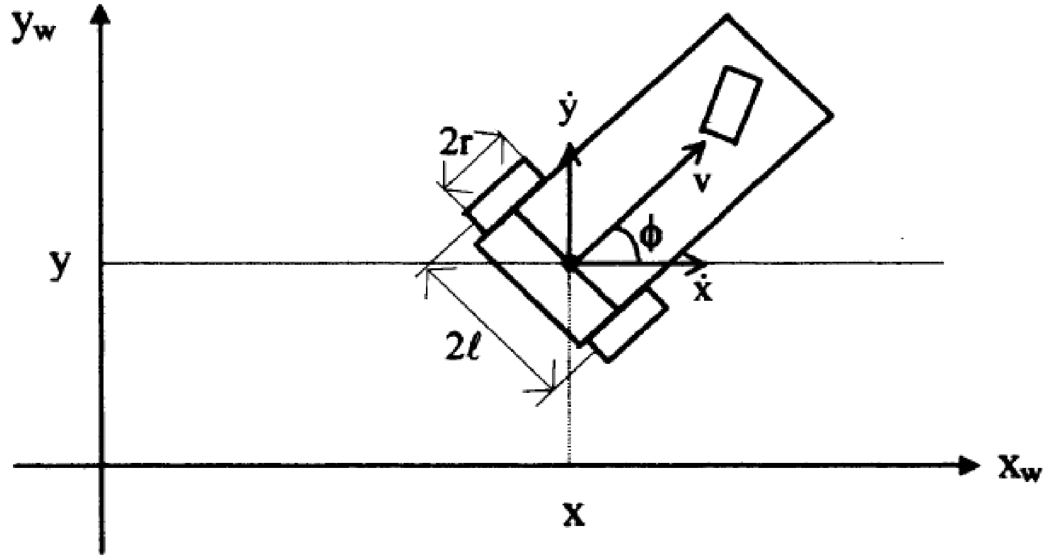


Fig. 1. Mobile robot configuration.

$$\lambda = -m\phi(\dot{x}\cos\phi + \dot{y}\sin\phi)$$

r: radius of the wheel
$2l$: length of the rear wheel axis
T_1 and T_2 :Torques provided by two motors attached to the rear wheels
m: Mass of the mobile robot
I: Moment of inertia of the mobile robot
λ: Lagrange multiplier

- The tracking problem consists of making the trajectory q of the mobile robot follow a reference trajectory q_{r}.

  The reference trajectory $q_{\mathrm{r}} = [x_{\mathrm{r}}(t) \quad y_{\mathrm{r}}(t) \quad \emptyset_{\mathrm{r}}(t)]$ is generated by a reference vehicle/ robot whose equations are

$$\dot{x}_{\mathrm{r}} = v_{\mathrm{r}} \cos \phi_{\mathrm{r}}$$
$$\dot{y}_{\mathrm{r}} = v_{\mathrm{r}} \sin \phi$$
$$\dot{\phi}_{\mathrm{r}} = \omega_{\mathrm{r}}$$

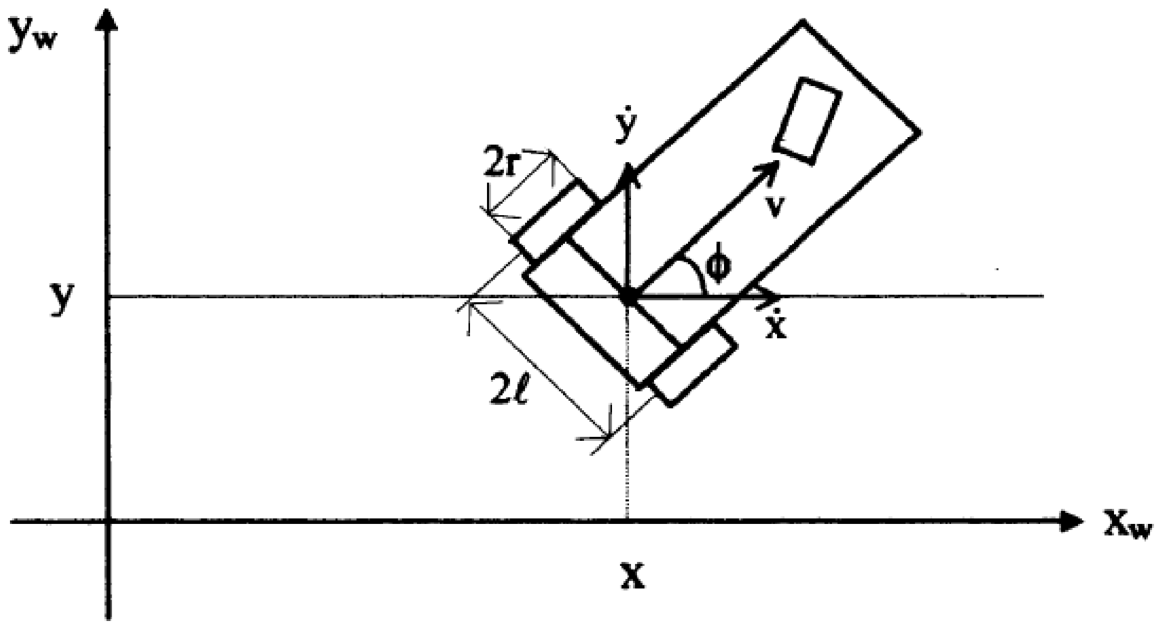


Fig. 1. Mobile robot configuration.

The subscript "r" stands for reference, and v_r and ω_r are the reference translational (linear) velocity and the reference rotational (angular) velocity, respectively. We assume that v_r and ω_r , as well as their derivatives are available and that they all are bounded.

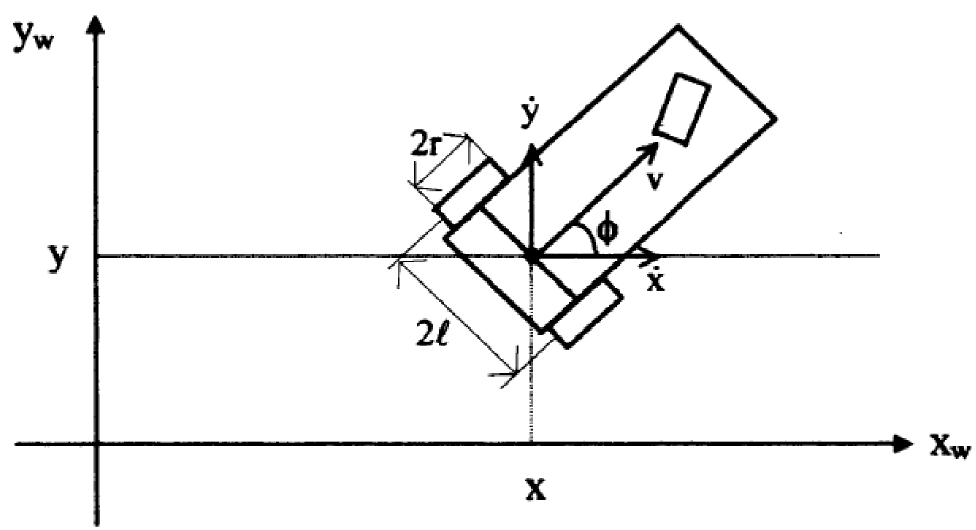
Assumption A1. For the tracking problem it is assumed that the reference velocities v_r and ω_r do not both go to zero simultaneously. That is, it is assumed that at any time either $\lim_{t\to\infty} v_r(t) \neq 0$ and/or $\lim\limits_{t\to\infty} \omega_r(t) \neq 0$

The tracking problem, under the **Assumption A1**, is to find a feedback control law

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u(q, \dot{q}, q_r, v_r, \omega_r, \dot{v}_r, \dot{\omega}_r)$$

$$\lim_{t\to\infty} \tilde{q}(t) = 0,$$

$$\tilde{q}(t) = q_r(t) - q(t)$$



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$$\lim_{t\to\infty} \tilde{q}(t) = 0,$$

$$\tilde{q}(t) = q_r(t) - q(t)$$

- We define the equivalent trajectory tracking error a:$e = T\tilde{q}$

$$\text{where } e = [e_1, e_2, e_3]^T, \text{ and } T = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Note that since $\boldsymbol{T}$ matrix is nonsingular, e is non zero as long as $\tilde{q} \neq 0$.

- Assuming that the angles $\emptyset_r$ and $\emptyset$ are given in the range $[-\pi, \pi]$, we have the equivalent trajectory tracking error e = 0 only if $q = q_r$.

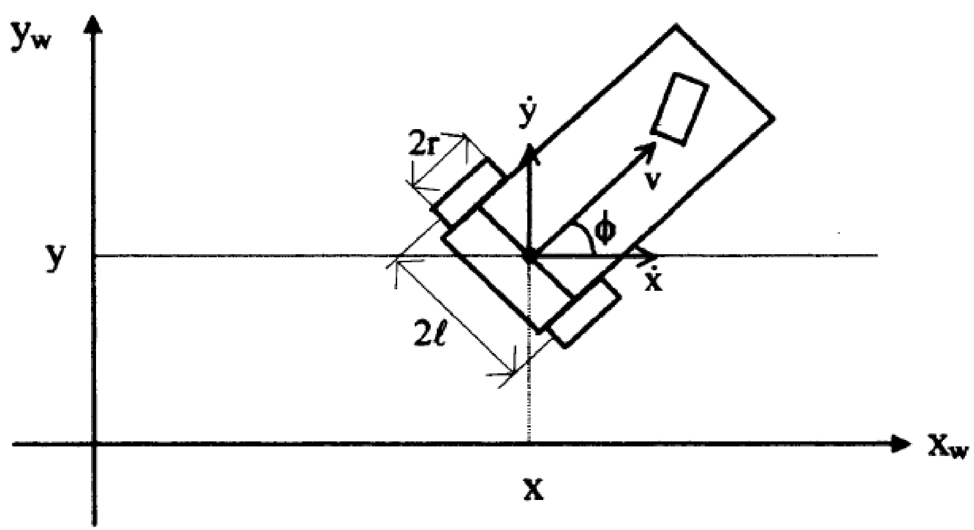


Fig. 1. Mobile robot configuration.

- Using the nonholonomic constraint, the derivative of the trajectory tracking error given in

$$\dot{x}\sin\phi - \dot{y}\cos\phi = 0$$

$$\text{where } e = [e_1, e_2, e_3]^{\mathrm{T}}, \text{ and } T = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \dot{e}_1 &= e_2\omega - v + v_{\mathrm{r}}\cos e_3 \\ \dot{e}_2 &= -e_1\omega + v_{\mathrm{r}}\sin e_3 \\ \dot{e}_3 &= \omega_{\mathrm{r}} - \omega \end{aligned}$$

$$v = \dot{x}\cos\phi + \dot{y}\sin\phi$$

$$\omega = \dot{\phi}$$

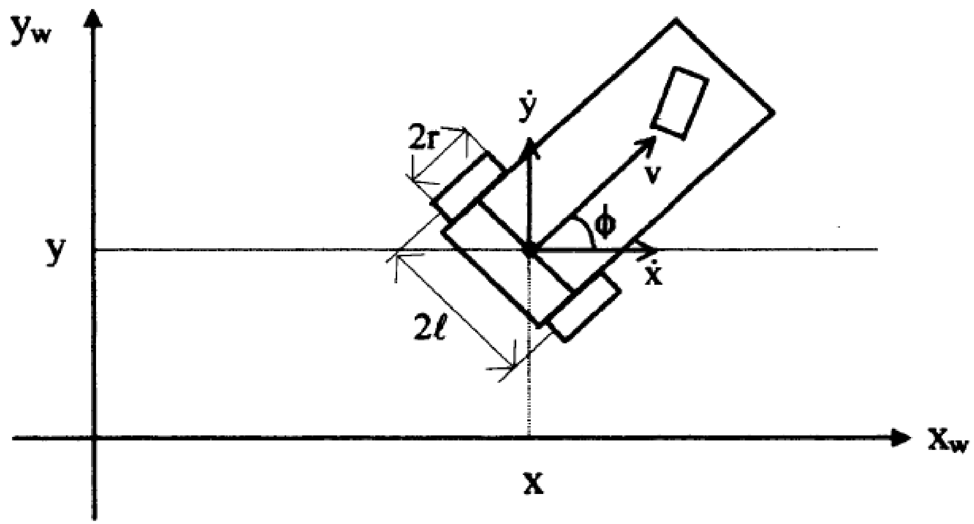


Fig. 1. Mobile robot configuration.

- **Tracking controller design**

Here, the goal is to design a controller to force the tracking error $e = [e_1 \quad e_2 \quad e_3]$ to zero. Using backstepping technique, since the actual control variables u_1 and u_2 do not appear in

$$\dot{e}_1 = e_2\omega - v + v_r \cos e_3$$
$$\dot{e}_2 = -e_1\omega + v_r \sin e_3$$
$$\dot{e}_3 = \omega_r - \omega$$

We consider variables v and ω as virtual controls. Let v_d and ω_d denote the desired virtual controls for the mobile robot. That is, with v_d and ω_d the trajectory tracking error e converges to zero asymptotically. Also let us define $\widetilde{v}$ and $\widetilde{\omega}$ as virtual control errors. Then, v and ω can be written as

$$v = v_d + \tilde{v}$$

$$\omega = \omega_d + \tilde{\omega}$$

Let us choose the virtual controls v_d and ω_d, as

$$v_d(v_r, \omega_r, e_1, e_3) = v_r \cos e_3 + k_1(v_r, \omega_r)e_1$$
$$\omega_d(v_r, \omega_r, e_2, e_3) = \omega_r + k_2 v_r e_2 + k_3(v_r, \omega_r) \sin e_3$$

- **Tracking controller design**

Let us choose the virtual controls v_d and ω_d, as

$$v_{\mathrm{d}}(v_{\mathrm{r}}, \omega_{\mathrm{r}}, e_1, e_3) = v_{\mathrm{r}} \cos e_3 + k_1(v_{\mathrm{r}}, \omega_{\mathrm{r}}) e_1$$
$$\omega_{\mathrm{d}}(v_{\mathrm{r}}, \omega_{\mathrm{r}}, e_2, e_3) = \omega_{\mathrm{r}} + k_2 v_{\mathrm{r}} e_2 + k_3(v_{\mathrm{r}}, \omega_{\mathrm{r}}) \sin e_3$$

where k_2 is a positive constant and k_1 and k_3 are bounded continuous functions with bounded first derivatives, strictly positive. Observe that our approach from here on is general for any v_d and ω_d (with well defined first derivatives), i.e. any differentiable control law that makes the kinematics model of the mobile robot track a desired trajectory can be used instead of

$$v_{\mathrm{d}}(v_{\mathrm{r}}, \omega_{\mathrm{r}}, e_1, e_3) = v_{\mathrm{r}} \cos e_3 + k_1(v_{\mathrm{r}}, \omega_{\mathrm{r}}) e_1$$
$$\omega_{\mathrm{d}}(v_{\mathrm{r}}, \omega_{\mathrm{r}}, e_2, e_3) = \omega_{\mathrm{r}} + k_2 v_{\mathrm{r}} e_2 + k_3(v_{\mathrm{r}}, \omega_{\mathrm{r}}) \sin e_3$$

Now consider the following adaptive control

$$u_1 = \hat{\beta}_1(-c_1\tilde{v} + e_1 + \dot{v}_{\mathrm{d}}) \qquad \dot{\hat{\beta}}_1 = -\gamma_1 \mathrm{sign}(b_1)\tilde{v}(-c_1\tilde{v} + e_1 + \dot{v}_{\mathrm{d}})$$
$$u_2 = \hat{\beta}_2\left(-c_2\tilde{\omega} + \frac{1}{k_2}\sin e_3 + \dot{\omega}_{\mathrm{d}}\right) \qquad \dot{\hat{\beta}}_2 = -\gamma_2 \mathrm{sign}(b_2)\tilde{\omega}\left(-c_2\tilde{\omega} + \frac{1}{k_2}\sin e_3 + \dot{\omega}_{\mathrm{d}}\right)$$

where c_1, c_2, γ_1, and γ_2 are positive constants and $\hat{\beta}_1$ is an estimate of $\beta_1 = 1/b_1$ and $\hat{\beta}_2$ is an estimate of $\beta_2 = 1/b_2$.

Result 1. *If Assumption A_1 holds, then the adaptive control* scheme *makes the origin* $e = 0$ *uniformly asymptotically stable.*

Proof. Consider the following Lyapunov function candidate

$$V_1 = \frac{1}{2}(e_1^2 + e_2^2) + \frac{1}{k_2}(1 - \cos e_3)$$

where k_2 is a positive constant. Clearly V_1 is positive definite and $V_1 = 0$ only if $e = 0$.

Taking the time derivative of V_1, we obtain

$$\dot{V}_1 = e_1(-v + v_r \cos e_3) + e_2 v_r \sin e_3 + \frac{1}{k_2}\sin e_3(\omega_r - \omega)$$

$$\dot{V}_1 = -k_1 e_1^2 - \frac{k_3}{k_2}\sin^2 e_3 - \tilde{v} e_1 - \tilde{\omega}\frac{1}{k_2}\sin e_3$$

we find the time derivatives of $\tilde{v}$ and $\tilde{\omega}$, as

$$\dot{\tilde{v}} = \dot{v} - \dot{v}_d = \ddot{x}\cos\phi - \dot{x}\sin\phi\dot{\phi} + \ddot{y}\sin\phi + \dot{y}\cos\phi\dot{\phi} - \dot{v}_d = b_1 u_1 - \dot{v}_d$$

$$\dot{\tilde{\omega}} = \dot{\omega} - \dot{\omega}_d = \ddot{\phi} - \dot{\omega}_d = b_2 u_2 - \dot{\omega}_d$$

Consider the Lyapunov function candidate

$$V_2 = V_1 + \frac{1}{2}(\tilde{v}^2 + \tilde{\omega}^2) + \frac{|b_1|}{2\gamma_1}\tilde{\beta}_1^2 + \frac{|b_2|}{2\gamma_2}\tilde{\beta}_2^2$$

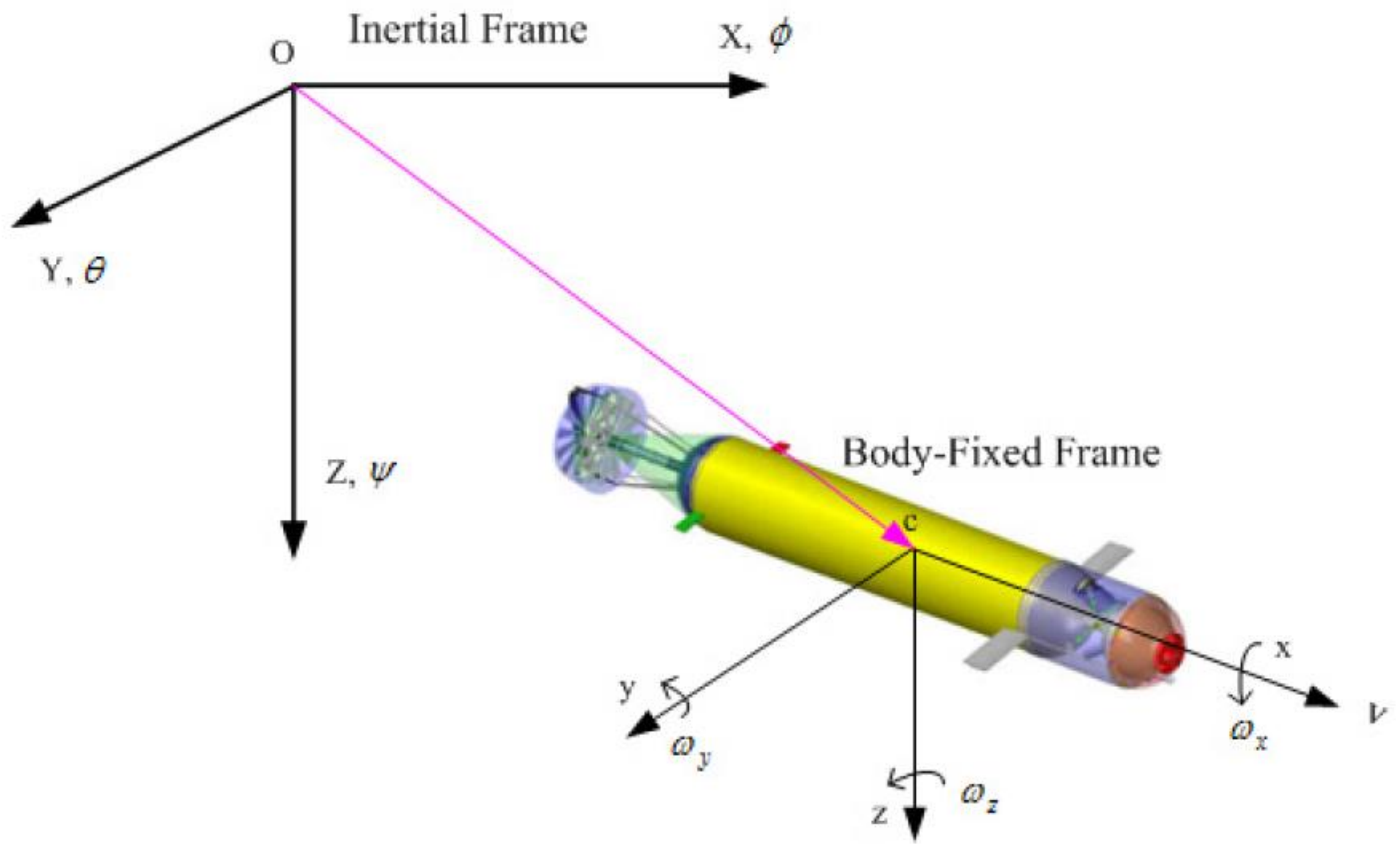
where $\tilde{\beta}_1 = \beta_1 - \hat{\beta}_1 = 1/b_1 - \hat{\beta}_1$ and $\tilde{\beta}_2 = \beta_2 - \hat{\beta}_2 = 1/b_2 - \hat{\beta}_2$.

$$\dot{V}_2 = -k_1 e_1^2 - \frac{k_3}{k_2}\sin^2 e_3 - c_1\tilde{v}^2 - c_2\tilde{\omega}^2 \leqslant 0$$

Since V_2 is bounded from below and $\dot{V}_2$ is negative semi-definite, V_2 converges to a finite limit. Also, V_2, as well as, e_1, e_2, e_3, $\tilde{v}$, $\tilde{\omega}$, $\hat{\beta}_1$, and $\hat{\beta}_2$ are all bounded.

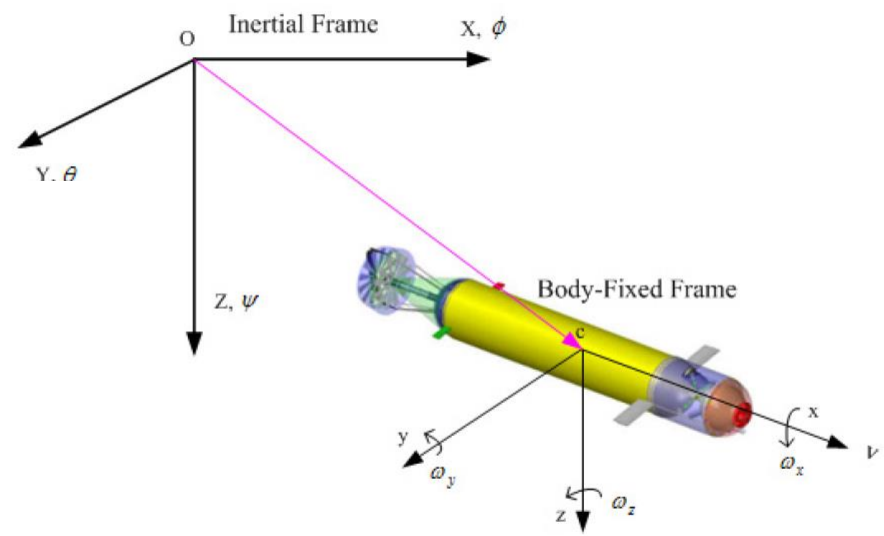
the second derivative of V_2 can be written as

$$\begin{aligned}\ddot{V}_2 = &-2k_1 e_1 e_2(\omega_\mathrm{r} + k_2 v_\mathrm{r} e_2 + k_3 \sin e_3 + \tilde{\omega}) + 2k_1 e_1(k_1 e_1 + \tilde{v}) - \dot{k}_1 e_1^2 \\ &+ \frac{2k_3}{k_2}\cos e_3 \sin e_3 (k_2 v_\mathrm{r} e_2 + k_3 \sin e_3 + \tilde{\omega}) - \frac{\dot{k}_3}{k_2}\sin^2 e_3 - 2c_1\tilde{v}(b_1\hat{\beta}_1(-c_1\tilde{v} + e_1 + \dot{v}_\mathrm{d}) - \dot{v}_\mathrm{d}) \\ &- 2c_2\tilde{\omega}\left(b_2\hat{\beta}_2\left(-c_2\tilde{\omega} + \frac{1}{k_2}\sin e_3 + \dot{\omega}_\mathrm{d}\right) - \dot{\omega}_\mathrm{d}\right)\end{aligned}$$

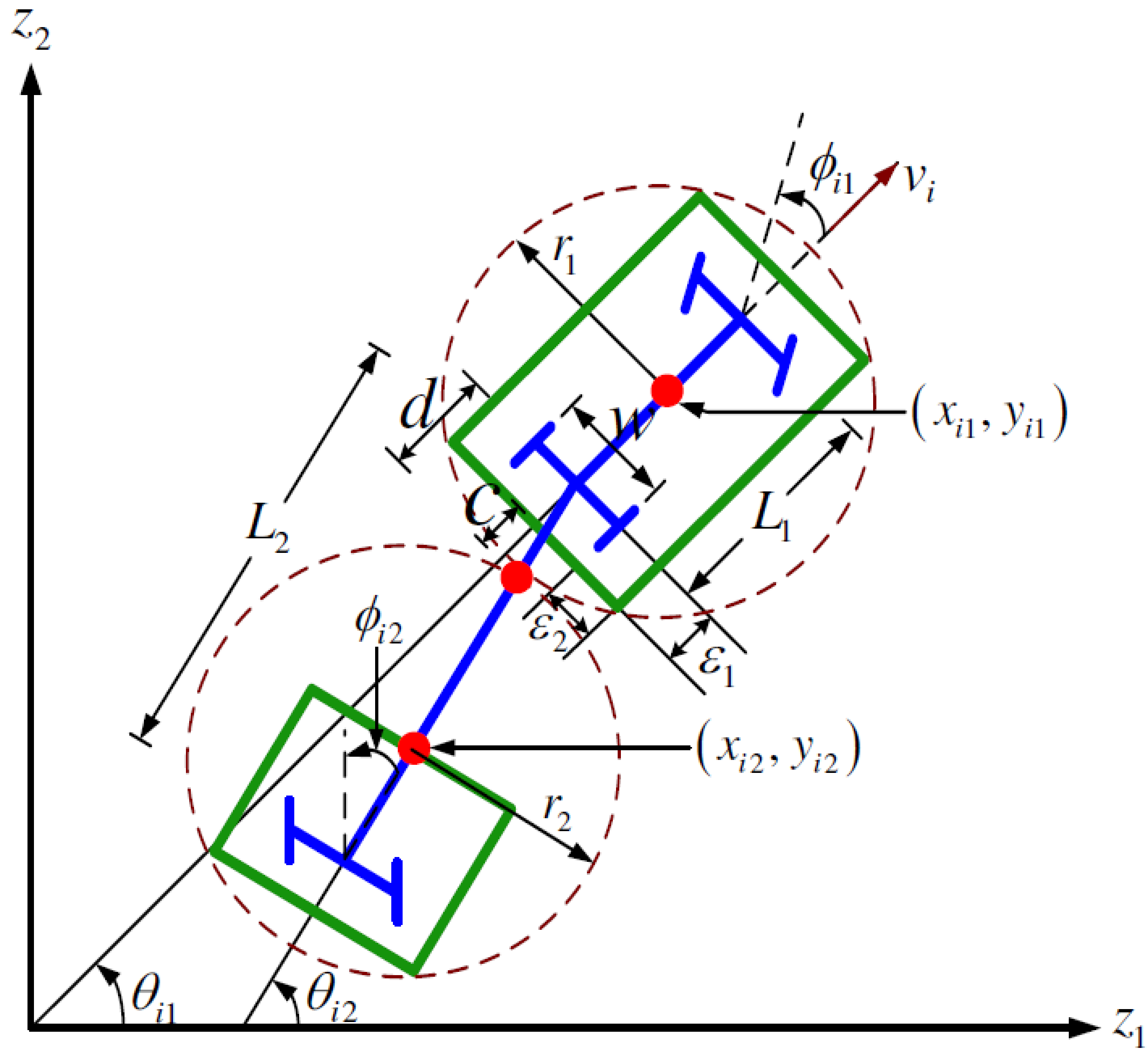


Inertial Frame
O
X, ϕ
Y, θ
Z, ψ
Body-Fixed Frame
c
x
ν
ω_x
y
ω_y
z
ω_z

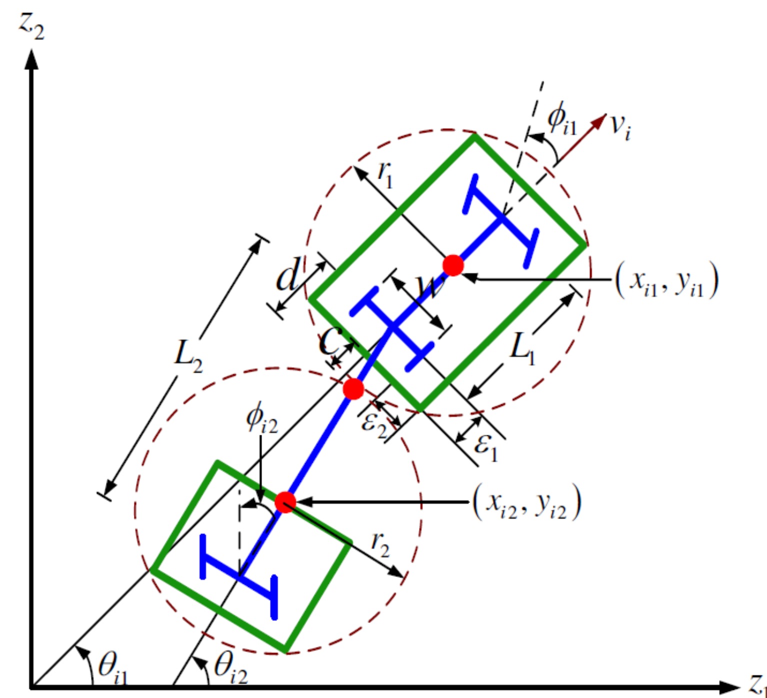
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v\cos\theta\cos\psi \\ v\cos\theta\sin\psi \\ -v\sin\theta \end{bmatrix},$$



$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$



z_2
z_1
ϕ_{i1}
v_i
r_1
d
w
(x_{i1}, y_{i1})
L_2
c
L_1
ε_2
ε_1
ϕ_{i2}
(x_{i2}, y_{i2})
r_2
θ_{i1}
θ_{i2}



$$
\begin{aligned}
x_{i2} &= x_{i1} - \frac{L_1}{2}\cos\theta_{i1} - \frac{L_2+2d}{2}\cos\theta_{i2},\\
y_{i2} &= y_{i1} - \frac{L_1}{2}\sin\theta_{i1} - \frac{L_2+2d}{2}\sin\theta_{i2}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_{i1} &= v_i\cos\theta_{i1} - \tfrac{L_1}{2}\omega_{i1}\sin\theta_{i1},\\
\dot{y}_{i1} &= v_i\sin\theta_{i1} + \tfrac{L_1}{2}\omega_{i1}\cos\theta_{i1},\\
\dot{\theta}_{i1} &= \tfrac{v_i}{L_1}\tan\phi_{i1} =: \omega_{i1},\\
\dot{\theta}_{i2} &= -\tfrac{1}{L_2}\sec\phi_{i2}\sin(\phi_{i2}-\theta_{i1}+\theta_{i2}) =: \omega_{i2},\\
\dot{v}_i &:= \sigma_{i1}, \qquad \dot{\omega}_{i1} := \sigma_{i2}, \qquad \dot{\omega}_{i2} := \sigma_{i3},
\end{aligned}
$$