Advanced Mechatronics Engineering

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- Sampled signals are easier to transmit.
- Transmitted signals can be regenerated without transmission error.
- Sampled signals can more easily be coded (cryptology).
- Sampled signals can better be modulated (multiplexing).

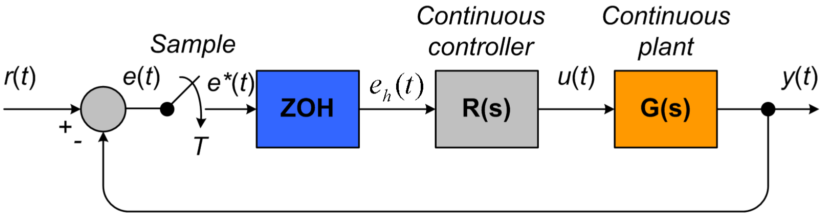


Figure: Sampling.

Types of Sampled Data Systems

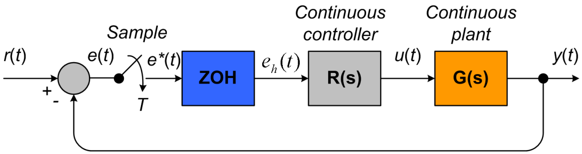
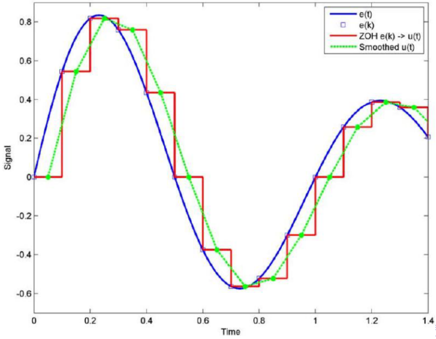


Figure: Sampling.



Zero Order Hold Sampling

- A ZOH samples the current signal and holds that value until the next sample.
- In most systems, it is difficult to generate and transmit narrow, large amplitude pulses
- We can often use a variety of filtering and interpolation techniques to reconstruct the original time-domain signal, however often the zero-order hold signal is sufficiently accurate.

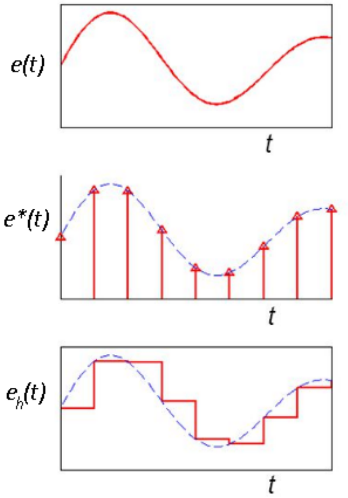


Figure: ZOH.

Sampling of Arbitrary Signals

Given $f(t) \Rightarrow$ what is $f^*(s)$?

Example

$$F(t) = \varepsilon(t) \quad \Rightarrow \quad F(s) = \frac{1}{s}$$

$$f^*(t) = \sum_{k=0}^{\infty} f(kt)\delta(t - kT) = \sum_{k=0}^{\infty} \delta(t - kT)$$

$$\begin{aligned} f^*(s) &= \sum_{k=0}^{\infty} \exp^{-kTs} = 1 + \exp^{-Ts} + \exp^{-2Ts} + \cdots \\ &= \frac{1}{1 - \exp^{-Ts}} \end{aligned}$$

Is there a simple way to compute $f^*(s)$ out of $f(s)$?

Is there a simple way to compute $F^*(s)$ out of $F(s)$?
For poles of $F(s)$ with multiplicity 1 (simple poles):

$$F(s) = \frac{P(s)}{Q(s)} \quad \Rightarrow \quad F^*(s) = \sum_{n=1}^{k} \frac{P(s_n)}{Q'(s_n)} \frac{1}{1 - \exp^{-T(s-s_n)}}$$

k is the number of poles.
Example: $F(s) = \frac{1}{s} \quad \Rightarrow$ One pole at zero
$\Rightarrow P(s) = 1 \ , \ Q(s) = s \ \Rightarrow \ Q'(s) = 1$

$$F^*(s) = \sum_{n=1}^{1} \frac{1}{1} \frac{1}{1 - \exp^{-T(s-0)}} = \frac{1}{1 - \exp^{-Ts}}$$

Sampling of Arbitrary Signals

$$F(s) = \frac{1}{s(s+1)} \quad \Rightarrow \quad \text{two poles, one at zero and ont at -1}$$

$$\begin{aligned}
F^*(s) &= \frac{P(s_1)}{Q'(s_1)} \frac{1}{1 - \exp^{-T(s-s_1)}} + \frac{P(s_2)}{Q'(s_2)} \frac{1}{1 - \exp^{-T(s-s_2)}} \\
&= \frac{1}{1} \frac{1}{1 - \exp^{-Ts}} + \frac{1}{-1} \frac{1}{1 - \exp^{-T(s+1)}} \\
&= \frac{1}{1 - \exp^{-Ts}} - \frac{1}{1 - \exp^{-T} \exp^{-Ts}} \\
&= \frac{1 - \exp^{-T} \exp^{-Ts} - 1 + \exp^{-Ts}}{(1 - \exp^{-Ts})(1 - \exp^{-T} \exp^{-Ts})} \\
&= \frac{(1 - \exp^{-T}) \exp^{-Ts}}{(1 - \exp^{-Ts})(1 - \exp^{-T} \exp^{-Ts})}
\end{aligned}$$

It turns out that, in both examples, $F^*(s)$ is a function of the term $\exp^{-Ts}$.

Sampling of Arbitrary Signals

The rule is pretty obvious and easy to derive. We use the partial fraction expansion on $F(s)$:

$$F(s) = \sum_{n=1}^{k} \frac{P(s_n)}{Q'(s_n)} \frac{1}{s - s_n}$$

when going from $F(s) \Rightarrow F^*(s)$, we need only to find $F^*(s)$ for $\frac{a}{s-b}$, and then we know the general formula for all $F(s)$ with simple poles

$$F(s) = \frac{a}{s-b} \quad \Rightarrow \quad F^*(s) = \frac{a}{1 - \exp^{-T(s-b)}}$$

as can be easily verified by looking at the infinite series.

Similar for multiple poles (multiplicity of poles n is m_n):

$$F^*(s) = \sum_{n=1}^{k} \sum_{i=1}^{m_n} \frac{(-1)^{m_n-1} K_{ni}}{(m_n-1)!} \frac{\partial^{m_n-i} \Delta T(s)}{\partial s^{m_n-i}} \Big|_{s=s-s_n}$$

where

$$K_{ni} = \frac{1}{(i-1)!} \frac{\partial^{i-1} \left[(s-s_n)^{m_n} F(s) \right]}{\partial s^{i-1}} \Big|_{s=s_n}$$

and

$$\Delta T(s) = \frac{1}{1 - \exp^{-Ts}}$$

Example

$$F(s) = \frac{2}{(s+a)^3} \quad \Rightarrow \quad s_1 = -a \;\; ; \;\; m_1 = 3$$

$$K_{11} = \frac{1}{1}(s+a)^3 F(s) \mid_{s=-a} = 2 \mid_{s=-a} = 2$$

$$K_{12} = \frac{1}{1}\frac{\partial}{\partial s}(2)|_{s=-a} = 0$$

$$K_{13} = \frac{1}{2}\frac{\partial^2 (2)}{\partial s^2} \mid_{s=-a} = 0$$

only the first term gives a contribution

$$F^*(s) = \frac{(-1)^2 2}{(3-1)!}\frac{\partial \Delta T(s)}{\partial s^2} \mid_{s=s-s_n}$$

Sampling of Arbitrary Signals

$$F^*(s) = \frac{(-1)^2 2}{(3-1)!}\frac{\partial \Delta T(s)}{\partial s^2}\Big|_{s=s-s_n}$$

$$\frac{\partial}{\partial s}\Delta T(s) = \frac{\partial}{\partial s}\left(\frac{1}{1-\exp^{-Ts}}\right) = \frac{-T\exp^{-Ts}}{(1-\exp^{-Ts})^2}$$

$$\frac{\partial^2}{\partial s^2}\Delta T(s) = \frac{\partial}{\partial s}\left(\frac{\partial}{\partial s}\Delta T(s)\right) = \frac{\partial}{\partial s}\left(\frac{-T\exp^{-Ts}}{(1-\exp^{-Ts})^2}\right)$$

$$\frac{\partial^2}{\partial s^2}\Delta T(s) = \frac{T^2\exp^{-Ts}(1+2\exp^{-Ts})}{(1-\exp^{-Ts})^3}$$

evaluate at $s = s - s_n = s + a$

$$F^*(s) = \frac{T^2 e^{-T(s+a)}(1+2e^{-T(s+a)})}{(1-e^{-T(s+a)})^3} = \frac{T^2 e^{-Ts}e^{-aT}(1+2e^{-aT}e^{-Ts})}{(1-e^{-aT}e^{-Ts})^3}$$

$$F^*(s) = \frac{T^2 e^{-T(s+a)}(1 + 2e^{-T(s+a)})}{(1 - e^{-T(s+a)})^3} = \frac{T^2 e^{-Ts} e^{-aT}(1 + 2e^{-aT} e^{-Ts})}{(1 - e^{-aT} e^{-Ts})^3}$$

Again we obtained an expression in e^{-Ts} **Let us set:**
$z = e^{Ts} \Leftrightarrow z^{-1} = e^{-Ts}$

$$F^*(s) = \widetilde{F}(z^{-1}) = \frac{T^2 z^{-1} e^{-aT}(1 + 2e^{-aT} z^{-1})}{(1 - e^{-aT} z^{-1})^3}$$

$$= F(z) = \frac{T^2 z e^{-aT}(z + 2e^{-aT})}{(z - e^{-aT})^3}$$

Sampling of Arbitrary Signals

Laplace Transform

$$f(t) \implies F(s) \quad \text{Laplace}$$

$$f(t) \implies \widetilde{F}(s^{-1}) \quad \text{Inverse Laplace}$$

Z-Transform

$$f(t) \implies F(z) \quad \text{Z-Transform}$$

$$f(t) \implies \widetilde{F}(z^{-1}) \quad \text{Inverse Z-Transform}$$

Like the Laplace transform, also the z-transform has a physical meaning:

Physical Meaning

$$
\begin{aligned}
s &= \text{Derivative } \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\
\frac{1}{s} = s^{-1} &= \text{Integration } \left(\int\right) \\
z &= \text{Left shifting by } T \\
\frac{1}{z} = z^{-1} &= \text{Right shifting by } T
\end{aligned}
$$

Sampling of Arbitrary Signals

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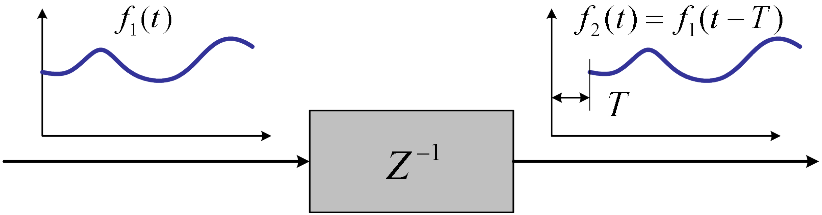


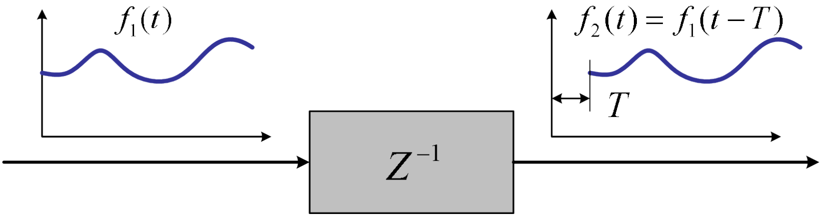


Figure: ZT.

The signal got delayed by T. Like in the Laplace transform, physically requires higher order denominator than nominators.

$$g(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \ldots + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_0} = \sum_{k=0}^{\infty} g_k z^{-k}$$

Comparison of the coefficients leads to the recursive formula:

$$g_k = b_{n-k} - a_{n-1} g_{k-1} - a_{n-2} g_{k-2} - \ldots - a_0 g_{k-n}$$

where b_k, g_k=0.

Example

$$g(z) = \frac{z^3 + 2z^2 + 3z}{z^4 - 1}$$

$$\begin{aligned}
g_0 &= b_4 - a_3 g_{-1} - \ldots = b_4 = 0 \\
g_1 &= b_3 - a_3 g_0 = 1 - 0 = 1 \\
g_2 &= b_2 - a_3 g_1 - a_2 g_0 = 2 - 0 - 0 = 2 \\
g_3 &= b_1 - a_3 g_2 - a_2 g_1 - a_1 g_0 = 3 - 0 - 0 - 0 = 3 \\
g_4 &= b_0 - a_3 g_3 - a_2 g_2 - a_1 g_1 - a_0 g_0
\end{aligned}$$

$$g(z) = \frac{z^3 + 2z^2 + 3z}{z^4 - 1} = \frac{y(z)}{u(z)}$$

$$z^4 y(z) - y(z) = z^3 u(z) + 2z^2 u(z) + 3zu(z)$$

Now apply the shifting property

$$y^*(t + 4T) - y^*(t) = u^*(t + 3T) + 2u^*(t + 2T) + 3u^*(t + T)$$

or

$$y^*(t + T) = y^*(t - 3T) + u^*(t) + 2u^*(t - T) + 3u^*(t - 2T)$$

Given a system with the z-transfer function $g(z)$m we can simulate the behaviour of the output out of measurements of previous inputs and outputs.

Given

$$g(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \ldots + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_0}$$

We can obtain

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{F}\mathbf{x}(k) + \mathbf{g}u(k) \\ y(k) &= \mathbf{h}\mathbf{x}(k) + iu(k) \end{aligned}$$

F: System matrix
g: Input vector
h: Output vector
i: Direct coupling.

Relation Between z-Domain and Frequency Domain

ω-Domain

$$H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k}$$

z-Domain

$$H(z) = \sum_{k=0}^{M} b_k z^k$$

Comparing the above we see that the connection s setting $z = e^{j\omega}$ in $H(z)$, i.e.,

$$H(e^{j\omega}) = H(Z)\mid_{z=e^{j\omega}}$$

The z-Plane and the Unit Circle

If we consider that z-plane, we see that $H(e^{j\omega})$ corresponds to evaluating $H(z)$ on the unit circle

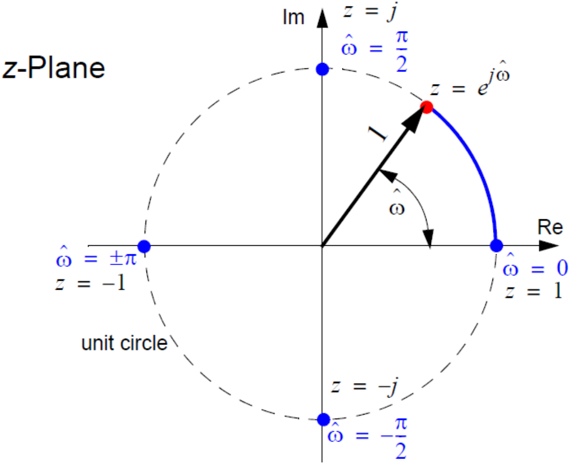
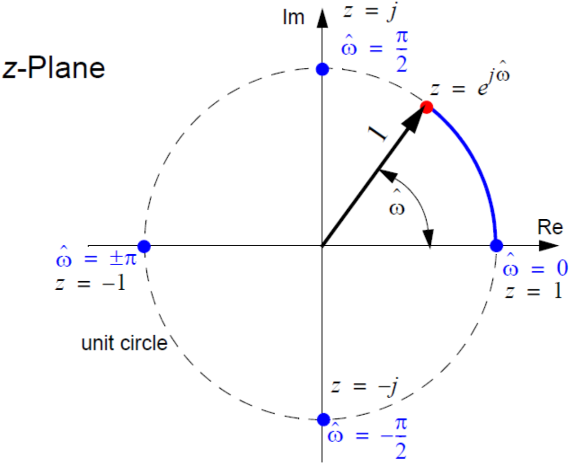


Figure: Z-plane.

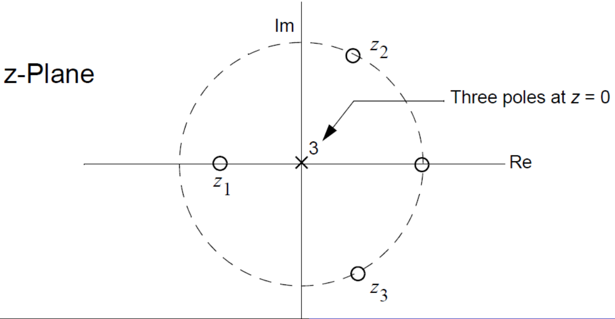
From this interpretation we also can see why $H(e^{j\omega})$ is periodic with period 2π. As ω increases it continues to sweep around the unit circle over and over again.



The Zeros and Poles of $H(z)$

$$H(Z) = \frac{(Z - Z_1)(Z - Z_2)(Z - Z_3)}{Z^3}$$

- The *zeros* are the locations where $H(z)$=0, i.e., z_1, z_2, z_3
- The *poles* are where $H(z) \to \infty$, i.e., $z \to 0$
- The *poles* and *zeros* only determine $H(z)$ for certain parameters
- A *pole-zero* displays the *pole* and *zero* locations in the z-plane



A method for study of linear constant discrete systems is:

- Compute the transfer function of the system $H(z)$.
- Compute the transform of the input signal, $E(z)$
- From the product, $E(z)H(z)$, which is the transform of the output signal, U.
- Invert the transform to obtain $u(kT)$

The Unit Impulse

The unit pulse is defined by

$$\begin{aligned} e_1(k) &= 1 \quad (k = 0) \\ &= 0 \quad (k \neq 0) \\ &= \delta_k. \end{aligned}$$

Therefore we have

$$E_1(z) = \sum_{-\infty}^{\infty} \delta_k z^{-k} = z^0 = 1$$

This result is much like the continuous case, wherein the Laplace transform of the unit impulse is the constant 1.0.
The quantity $E_1(z)$ gives us an instantaneous method of relate signals to systems: to characterize the system $H(z)$, consider the signal $u(k)$, which is the unit impulse response, then $U(z) = H(z)$.

The Unit Step

Consider the unit step function defined by

$$\begin{aligned} e_2(k) &= 1 \quad (k \geq 0) \\ &= 0 \quad (k < 0) \\ &= 1(k). \end{aligned}$$

In this case, the z-transform is

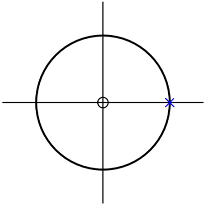
$$\begin{aligned} E_2(z) &= \sum_{k=-\infty}^{\infty} e_2(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k} \\ &= \frac{1}{1 - z^{-1}} \quad (\mid z^{-1} \mid < 1) \\ &= \frac{z}{z - 1} \quad (\mid z \mid > 1) \end{aligned}$$

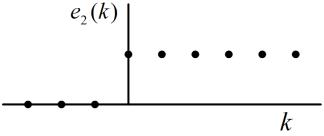
Here the transform is characterized by a zero at $z = 0$ and a pole at $z = 1$.

The Unit Step

$$\begin{aligned} E_2(z) &= \sum_{k=-\infty}^{\infty} e_2(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k} \\ &= \frac{1}{1 - z^{-1}} \qquad (\mid z^{-1} \mid < 1) \\ &= \frac{z}{z-1} \qquad (\mid z \mid > 1) \end{aligned}$$

The Laplace transform of the unit step is $1/s$; we may this keep in mind that a pole at $s = 0$ for a continuous signal corresponds in some way to a pole at $z = 1$ for discrete signals.





Exponential

The one-sided exponential in time is

$$\begin{aligned} e_3(k) &= r^k \quad (k \geq 0) \\ &= 0 \quad (k < 0) \end{aligned}$$

Now we get

$$\begin{aligned} E_3(z) &= \sum_{k=0}^{\infty} r^k z^{-k} \\ &= \sum_{k=0}^{\infty} \left(rz^{-1}\right)^k \\ &= \frac{1}{1 - rz^{-1}} \quad (\mid rz^{-1} \mid < 1) \\ &= \frac{z}{z - r} \quad (\mid z \mid > \mid r \mid) \end{aligned}$$

$$E_3(z) = \frac{1}{1 - rz^{-1}} \quad (\mid rz^{-1} \mid < 1)$$
$$= \frac{z}{z - r} \quad (\mid z \mid > \mid r \mid)$$

The pole of $E_3(z)$ is at $z = r$. We also know that $e_3(k)$ grows without bound if $\mid r \mid > 1$. We conclude that a z-transform that converges for large z and has a real pole outside the circle $\mid z \mid = 1$ corresponds to a growing signal.

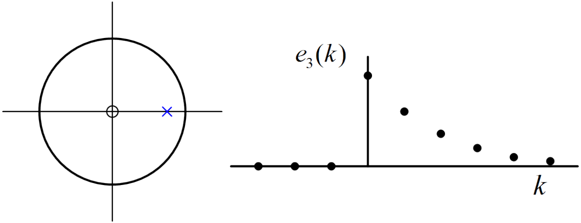


Figure: Exponential - $e_3(k)$ for the stable value $r = 0.6$

Consider the sinusoid $e_4(k) = [r^k \cos(k\theta)]1(k)$, where we assume $r \geq 0$. We can decompose $e_4(k)$ into the sum of two complex exponentials as

$$e_4(k) = r^k \left(\frac{e^{jk\theta} + e^{-jk\theta}}{2} \right) 1(k)$$

and because the z-transform is linear, we need only to compute the transform of each signal complex exponential and add the results later. We thus take first

$$e_5(k) = r^k e^{jk\theta} 1(k)$$

and compute

$$\begin{aligned} E_5(z) &= \sum_{k=0}^{\infty} r^k e^{jk\theta} z^{-k} \\ &= \sum_{k=0}^{\infty} \left(re^{j\theta} z^{-1} \right)^k \end{aligned}$$

General Sinusoid

$$
\begin{aligned}
E_5(z) &= \sum_{k=0}^{\infty} r^k e^{jk\theta} z^{-k} \\
&= \sum_{k=0}^{\infty} \left(re^{j\theta} z^{-1} \right)^k \\
&= \frac{1}{1 - re^{j\theta} z^{-1}} \\
&= \frac{z}{z - re^{j\theta}} \qquad (\mid z \mid > r)
\end{aligned}
$$

The signal $e_5(k)$ grows without bound as k gets large if and only if $r > 1$, and a system with this pulse response is BIBO stable if and only if $\mid r \mid < 1$. The bundary of stability is the unit circle.

To complete the argument given before $e_4(k) = r^k \cos(k\theta)1(k)$, we see immediately that the other half is found by replacing θ by $-\theta$

$$\mathcal{Z}\{r^k e^{-j\theta k} 1(k)\} = \frac{z}{z - re^{-j\theta}} \quad (|\, z \,| > r)$$

and thus

$$\begin{aligned} E_4(z) &= \frac{1}{2}\left(\frac{z}{z - re^{j\theta}} + \frac{z}{z - re^{-j\theta}}\right) \\ &= \frac{z(z - r\cos\theta)}{z^2 - 2r(\cos\theta)z + r^2} \quad (|\, z \,| > r) \end{aligned}$$

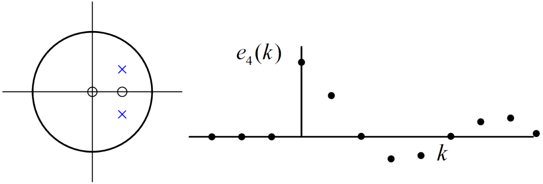


Figure: Exponential - $e_4(k)$ for $r = 0.7$ and $\theta = 45$ deg

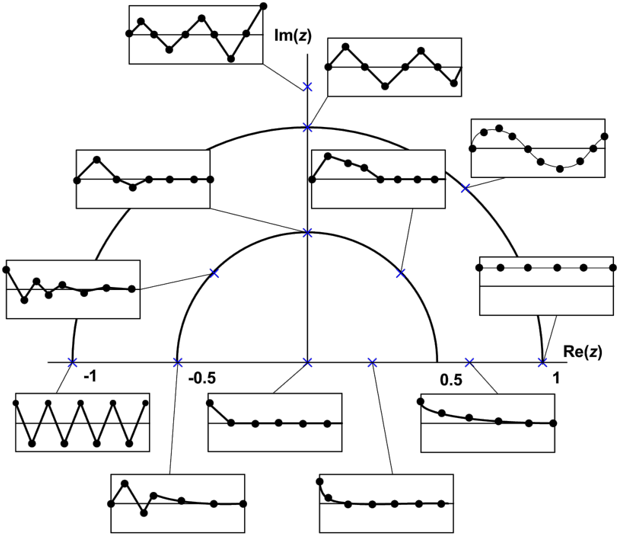


Figure: z-plane

Thanks

Questions please