Robotics

Islam S. M. Khalil

German University in Cairo

Inertia Vectors and Inertia Scalars

If S is a set of particles $P_1, \ldots, P_v$ of masses $m_1, \ldots, m_v$, respectively, $\mathbf{p}_i$ is the position vector from a point O to P_i $(i = 1, \ldots, v)$, and $\mathbf{n}_a$ is a unit vector, then a vector $\mathbf{I}_a$, called the inertia vector of S relative to O for $\mathbf{n}_a$, is defined as

$$\mathbf{I}_a \triangleq \sum_{i=1}^{v} m_i \mathbf{p}_i \times (\mathbf{n}_a \times \mathbf{p}_i) \tag{1}$$

A scalar I_{ab}, called the inertia scalar of S relative to O for $\mathbf{n}_a$ and $\mathbf{n}_b$, where $\mathbf{n}_b$ is a unit vector, is defined as

$$I_{ab} \triangleq \mathbf{I}_a \cdot \mathbf{n}_b \tag{2}$$

It follows from (1) and (2) that I_{ab} can be expressed as

$$\mathbf{I}_{ab} = \sum_{i=1}^{v} m_i \left(\mathbf{p}_i \times \mathbf{n}_a\right) \cdot \left(\mathbf{p}_i \times \mathbf{n}_b\right) \tag{3}$$

$$\mathbf{I}_{ab} = \sum_{i=1}^{v} m_i \left(\mathbf{p}_i \times \mathbf{n}_a\right) \cdot \left(\mathbf{p}_i \times \mathbf{n}_b\right) \quad (4)$$

This shows that

$$I_{ab} = I_{ba} \quad (5)$$

When $\mathbf{n}_a \neq \mathbf{n}_b$, I_{ab} is called the product of inertia of S relative to O for $\mathbf{n}_a$ and $\mathbf{n}_b$. When $\mathbf{n}_a = \mathbf{n}_b$, the corresponding inertia scalar sometimes is denoted by I_a (rather than by I_{aa}) and is called the moment of inertia of S with respect to line L_a, where is the line passing through point O and parallel to $\mathbf{n}_a$. The moment of inertia of S with respect to a line L_a can always be expressed as

$$\mathbf{I}_a = \sum_{i=1}^{v} m_i l_i^2 \quad (6)$$

where l_i^2 is the distance from P_i to line L_a, and as

$$\mathbf{I}_a = m k_a^2 \quad (7)$$

$$\mathbf{I}_a = mk_a^2 \tag{8}$$

where m is the total mass of S, and k_a is a real, non-negative quantity called the radius of gyration of S with respect to line L_a. Knowledge of the inertia vectors $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$ of a body B relative to a point O for three mutually perpendicular unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ enables on to find $\mathbf{I}_a$, the inertia vector of B relative to O for any unit vector $\mathbf{n}_a$, for

$$\mathbf{I}_a = \sum_{j=1}^{3} a_j \mathbf{I}_j \tag{9}$$

where a_1, a_2, and a_3 are defined as

$$a_j \triangleq \mathbf{n}_a \cdot \mathbf{n}_j \qquad (j = 1, 2, 3) \tag{10}$$

Inertia Vectors and Inertia Scalars

Similarly, I_{ab}, the inertial scalar of B relative to O for $\mathbf{n}_a$ and $\mathbf{n}_b$, can be found easily when the inertia scalars I_{jk} $(j, k = 1, 2, 3)$ are known, for

$$I_{ab} = \sum_{j=1}^{3} \sum_{k=1}^{3} a_j I_{jk} b_k \qquad (11)$$

where

$$b_k \triangleq \mathbf{n}_b \cdot \mathbf{n}_k \qquad (k = 1, 2, 3) \qquad (12)$$

The inertia scalars I_{jk} of a set S of particles relative to a point O for unit vectors $\mathbf{n}_j$ and $\mathbf{n}_k$ $(j, k = 1, 2, 3)$ can be used to define a square matrix $\mathbf{I}$, called the inertia matrix of $\mathbf{S}$ relative to O for $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ as follows

$$\mathbf{I} \triangleq \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \qquad (13)$$

The Dynamic Model provides the relations between the generalized forces **U** acting on the robot and the motion over time. It is governed by

$$\mathbf{\Phi}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \mathbf{U} \tag{14}$$

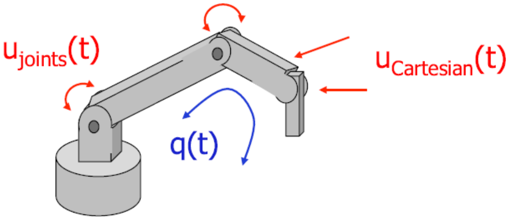


Figure: Dynamic model is a system of 2nd order differential equations.

Euler-Lagrange Method (Energy-Based Approach)

The basic assumption is that the robot consists of N links and considered as rigid bodies.

$$\frac{\partial \mathcal{L}}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}, t) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}, t)\right) = u_i \quad \textit{for} \quad i = 1, \cdots, N, \quad (15)$$

where $\mathcal{L}$ is the *Lagrangian* function. Further, q_i is the generalized coordinates of the ith degree-of-freedom, respectively.

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}) - U(\mathbf{q}) \quad (16)$$

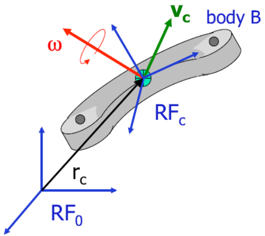
where $T(\mathbf{q}, \dot{\mathbf{q}})$ is the kinetic energy and $U(\mathbf{q})$ is the potential energy of the robot. u_i are the ith non-conservative force or the generalized forces performing work on q_i

$$m = \int_B \rho(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_B \mathrm{d}m \tag{17}$$

where m is the mass of body B and $\rho(x, y, z)$ is the mass density. The position of center of mass (CoM) is given by

$$\mathbf{r}_{\mathrm{c}} = \frac{1}{m}\int_B \mathbf{r}\mathrm{d}m \tag{18}$$

when all vectors are referred to a body frame RF_{c} attached to the center of mass, then $\mathbf{r}_{\mathrm{c}} \Rightarrow \int_B \mathbf{r}\mathrm{d}m=0$



Kinetic energy of a rigid body

$$\int_B \mathbf{r}\mathrm{d}m = 0 \;\Rightarrow\; T = \frac{1}{2}\int_b \mathbf{V}^{\mathrm{T}}(x,y,z)\mathbf{V}(x,y,z)\mathrm{d}m$$

We can use the fundamental kinematic relation

$$\mathbf{V} = \mathbf{V}_{\mathrm{c}} + \omega \times \mathbf{r} = \mathbf{V}_{\mathrm{c}} + \mathbf{S}(\omega)\mathbf{r}$$

where $\mathbf{S}(\omega)$ is a skew-symmetric matrix.

$$T = \frac{1}{2}\int_b \left(\mathbf{V}_{\mathrm{c}} + \mathbf{S}(\omega)\mathbf{r}\right)^{\mathrm{T}}\left(\mathbf{V}_{\mathrm{c}} + \mathbf{S}(\omega)\mathbf{r}\right)\mathrm{d}m$$

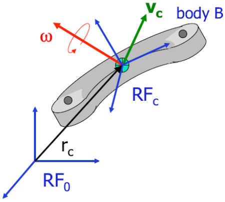


Figure: Kinetic Energy of body B.

$$T = \underbrace{\frac{1}{2}\int_b \mathbf{V}_{\mathrm{c}}^{\mathrm{T}}\mathbf{V}_{\mathrm{c}}\mathrm{d}m}_{t} + \underbrace{\int_B \mathbf{V}_{\mathrm{c}}^{\mathrm{T}}\mathbf{S}(\omega)\mathbf{r}\mathrm{d}m}_{0} + \underbrace{\int_B \mathbf{r}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}(\omega)\mathbf{S}(\omega)\mathbf{r}\mathrm{d}m}_{rotational\ \ kinetic\ \ energy}$$

Kinetic energy of a rigid body

$$T = \underbrace{\frac{1}{2}\int_b \mathbf{V}_\mathrm{c}^\mathrm{T}\mathbf{V}_\mathrm{c}\mathrm{d}m}_{t} + \underbrace{\int_B \mathbf{V}_\mathrm{c}^\mathrm{T}\mathbf{S}(\omega)\mathbf{r}\mathrm{d}m}_{0} + \underbrace{\int_B \mathbf{r}^\mathrm{T}\mathbf{S}^\mathrm{T}(\omega)\mathbf{S}(\omega)\mathbf{r}\mathrm{d}m}_{rotational\ \ kinetic\ \ energy}$$

$$T = \frac{1}{2}m\mathbf{V}_\mathrm{c}^\mathrm{T}\mathbf{V}_\mathrm{c} + \omega^\mathrm{T}\mathbf{I}_\mathrm{c}\omega$$

where $\mathbf{I}_\mathrm{c}$ is the body inertial matrix around CoM.

$$T = \sum_{i=1}^{N} T_i$$

$$T_i = \frac{1}{2}m_i\mathbf{v}_{\mathrm{c}i}^\mathrm{T}\mathbf{v}_{\mathrm{c}i} + \omega_i^\mathrm{T}\mathbf{I}_{\mathrm{c}i}\omega_i$$

$\mathbf{v}_{\mathrm{c}i}$ is the velocity of the CoM and ω_i is the angular velocity of the whole body.

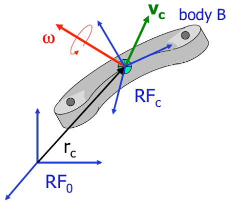


Figure: Kinetic Energy of body B.

Kinetic energy of a rigid body

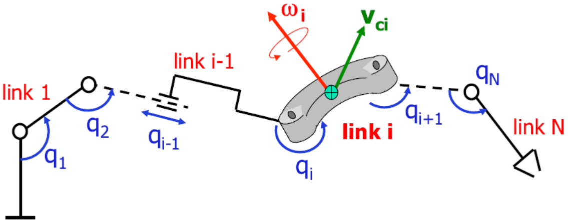


Figure: Kinetic Energy of body B.

$$\mathbf{v}_{ci} = \mathrm{J}_{\mathrm{Li}}(\mathbf{q})\dot{\mathbf{q}} \qquad \omega = \mathrm{J}_{\mathrm{Ai}}(\mathbf{q})\dot{\mathbf{q}}$$

$$\begin{aligned} T &= \frac{1}{2}\sum_{i=1}^{N}\left(m_i \mathbf{v}_{ci}^{\mathrm{T}}\mathbf{v}_{ci} + \omega_i^{\mathrm{T}}\mathbf{I}_{ci}\omega_i\right) \\ &= \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\left(\sum_{i=1}^{N} m_i \mathrm{J}_{\mathrm{Li}}^{\mathrm{T}}(\mathbf{q})\mathrm{J}_{\mathrm{Li}}(\mathbf{q}) + \mathrm{J}_{\mathrm{Ai}}^{\mathrm{T}}(\mathbf{q})\mathbf{I}_{ci}\mathrm{J}_{\mathrm{Ai}}(\mathbf{q})\right)\dot{\mathbf{q}} \quad (19) \end{aligned}$$

Kinetic energy of a rigid body

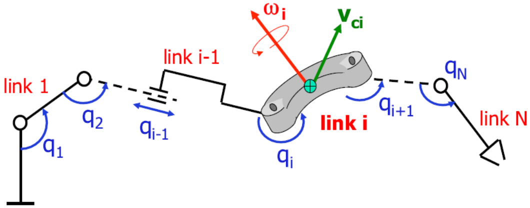


Figure: Kinetic Energy of body B.

$$T(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}} \underbrace{\left(\sum_{i=1}^{N} m_i \mathrm{J}_{\mathrm{Li}}^{\mathrm{T}}(\mathbf{q})\mathrm{J}_{\mathrm{Li}}(\mathbf{q}) + \mathrm{J}_{\mathrm{Ai}}^{\mathrm{T}}(\mathbf{q})\mathbf{I}_{ci}\mathrm{J}_{\mathrm{Ai}}(\mathbf{q})\right)}_{\mathbf{M}(\mathbf{q})} \dot{\mathbf{q}}$$

$$T(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$$

where $\mathbf{M}(\mathbf{q})$ is the robot inertia matrix (symmetric and positive definite for all configurations).

Potential energy of a rigid body

The potential enery of N rigid bodies is given by

$$U = \sum_{i=1}^{N} U_i$$

where the ith potential energy is

$$U_i = -m_i \mathbf{g}^{\mathrm{T}} \mathbf{r}_{0,\mathrm{c}i}$$

where $\mathbf{g}$ is the gravity acceleration vector. $\mathbf{r}_{0,\mathrm{c}i}$ is the position vector of the CoM of the ith link and depend on $\mathbf{q}$

$$\begin{pmatrix} \mathbf{r}_{0,\mathrm{c}i} \\ 1 \end{pmatrix} = {}^{0}\mathbf{A}_1(q_1)\,{}^{1}\mathbf{A}_2(q_2)\cdots{}^{i-1}\mathbf{A}_i(q_i) \begin{pmatrix} \mathbf{r}_{i,\mathrm{c}i} \\ 1 \end{pmatrix}$$

Applying Euler-Lagrange Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_k}}(\mathbf{q},\dot{\mathbf{q}},t)\right)-\frac{\partial \mathcal{L}}{\partial q_k}(\mathbf{q},\dot{\mathbf{q}},t)=u_k \quad \textit{for} \quad k=1,\cdots,N,$$

where $\mathcal{L}$ is the *Lagrangian* function. Further, q_i is the generalized coordinates of the ith degree-of-freedom, respectively.

$$\begin{aligned}\mathcal{L}(\mathbf{q},\dot{\mathbf{q}},t) &= T(\mathbf{q},\dot{\mathbf{q}})-U(\mathbf{q})\\ &= \frac{1}{2}\sum_{i,j} b_{ij}(q)\dot{q}_i\dot{q}_j-U(q)\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q_k}}(\mathbf{q},\dot{\mathbf{q}},t)=\sum_j b_{kj}\dot{q}_j\,,\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_k}}(\mathbf{q},\dot{\mathbf{q}},t)\right)=\sum_j b_{kj}\ddot{q}_j+\sum_{i,j}\frac{\partial b_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

$$\frac{\partial \mathcal{L}}{\partial q_k}(\mathbf{q},\dot{\mathbf{q}},t)=\frac{1}{2}\sum_{i,j}\frac{\partial b_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j-\frac{\partial U}{\partial q_k}$$

Applying Euler-Lagrange Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}(\mathbf{q},\dot{\mathbf{q}},t)\right) = \sum_j \underbrace{b_{kj}}_{nonlinear} \underbrace{\ddot{q}_j}_{linear} + \sum_{i,j} \underbrace{\frac{\partial b_{kj}}{\partial q_i}}_{nonlinear} \underbrace{\dot{q}_i \dot{q}_j}_{quadratic}$$

$$\frac{\partial \mathcal{L}}{\partial q_k}(\mathbf{q},\dot{\mathbf{q}},t) = \frac{1}{2}\sum_{i,j} \underbrace{\frac{\partial b_{ij}}{\partial q_k}}_{nonlinear} \underbrace{\dot{q}_i \dot{q}_j}_{quadratic} - \underbrace{\frac{\partial U}{\partial q_k}}_{nonlinear}$$

Dependence on q

- Linear terms in acceleration
- Quadratic terms in velocity
- Nonlinear terms in configuration

Applying Euler-Lagrange Equations

$$\underbrace{\sum_{j} b_{kj}\ddot{q}_j}_{Inertial\ \ tems} + \underbrace{\sum_{i,j} C_{kij}(q)\dot{q}_i\dot{q}_j}_{Centrifugal\ \ and\ \ Coriolis\ \ terms} + \underbrace{\frac{\partial U}{\partial q_k}}_{Gravity} = u_k \quad (k = 1, \cdots, N)$$

Robot Dynamic Model

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \mathbf{U}$$

Robot Dynamic Model

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{S}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{U}$$

$$\left(\dot{\mathbf{M}} - 2\mathbf{S}\right)^{\mathrm{T}} = -\left(\dot{\mathbf{M}} - 2\mathbf{S}\right)$$

Thanks

Questions please