Modeling of a Pantograph Haptic Device

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Kinematics of the Pantograph Haptic Device

The pantograph haptic device consists of 5 links with lengths l_i for $i = 1, \ldots, 5$. First, we assign frames of reference to each link, as shown in Fig. 1. The holonomic constrain of the pantograph mechanisms is given by

$$l_1\mathbf{a}_1 + l_2\mathbf{b}_1 - l_3\mathbf{c}_1 - l_4\mathbf{d}_1 - l_0\mathbf{n}_1 = 0, \tag{1}$$

where l_i for $i = 1, \ldots, 5$ is the length of links N, A, B, C, and D, respectively. We assume that link N is the frame of reference and calculate the rotation matrices of each link with respect to frame N as follows:

$$^{\mathrm{N}}\mathbf{R}^{\mathrm{A}} = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2}$$

where $^{\mathrm{N}}\mathbf{R}^{\mathrm{A}}$ is the rotation matrix of link A with respect to the frame of reference N, and q_1 is the generalized coordinate of the link A. Similarly, the rotation matrix ($^{\mathrm{N}}\mathbf{R}^{\mathrm{B}}$) of link B with respect to the frame of reference N is given by

$$^{\mathrm{N}}\mathbf{R}^{\mathrm{B}} = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

where q_2 is the generalized coordinate of link B. Rotation matrices of bodies C ($^{\mathrm{N}}\mathbf{R}^{\mathrm{C}}$) and D ($^{\mathrm{N}}\mathbf{R}^{\mathrm{D}}$) with respect to N are given respectively using

$$^{\mathrm{N}}\mathbf{R}^{\mathrm{C}} = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad {}^{\mathrm{N}}\mathbf{R}^{\mathrm{D}} = \begin{bmatrix} \cos q_4 & -\sin q_4 & 0 \\ \sin q_4 & \cos q_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

where q_3 and q_4 are the generalized coordinates of links C and D, respectively. All rotations of our pantograph haptic device are in-plane. Therefore, the angular velocities of the links are calculated by inspection as follows:

$$^{\mathrm{N}}\omega^{\mathrm{A}} = \dot{q}_1\mathbf{n}_3, \tag{5}$$

where $^{\mathrm{N}}\omega^{\mathrm{A}}$ is the angular velocity of link A with respect to the frame of reference N, and $\dot{q}_1$ is the time-derivative of q_1. Similarly, the angular velocities of links B, C, and D are given by

$$^{\mathrm{N}}\omega^{\mathrm{B}} = \dot{q}_2\mathbf{n}_3\ ,\ {}^{\mathrm{N}}\omega^{\mathrm{C}} = \dot{q}_3\mathbf{n}_3\ , \text{and } {}^{\mathrm{N}}\omega^{\mathrm{D}} = \dot{q}_4\mathbf{n}_3, \tag{6}$$

where $^{\mathrm{N}}\omega^{\mathrm{B}}$, $^{\mathrm{N}}\omega^{\mathrm{C}}$, and $^{\mathrm{N}}\omega^{\mathrm{D}}$ are the angular velocities of links B, C, and D with respect to the frame of reference N, respectively. $\mathbf{n}_3$ is a unit vector perpendicular to the page. Now we calculate the relation between the task space ($\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}^{\mathrm{T}}$) and joint space ($\mathbf{q} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}^{\mathrm{T}}$) of the pantograph haptic device, where x and y are the Cartesian coordinates of point e with respect to the frame of reference N, as shown in Fig. 1.

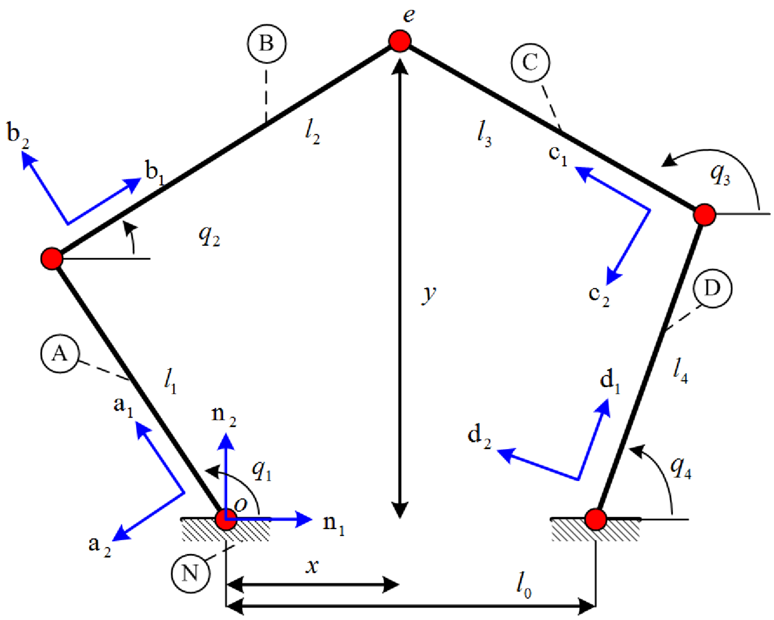

Figure 1. Kinematics of the pantograph haptic device. The position of the end-effector (point of interaction between the operator and the pantograph) is denoted using e, and is represented using position vector $\mathbf{r}^{oe}$.

0.1 Configuration Level Kinematics

In order to solve for x and y using q_1 and q_4 (the active angles of the pantograph haptic device), we define the following vector:

$$\begin{aligned} \mathbf{r}^{oe} &= x\mathbf{n}_1 + y\mathbf{n}_2, && (7)\\ &= l_1\mathbf{a}_1 + l_2\mathbf{b}_1, && (8)\\ &= l_1\left(\cos q_1\mathbf{n}_1 + \sin q_1\mathbf{n}_2\right) + l_2\left(\cos q_2\mathbf{n}_1 + \sin q_2\mathbf{n}_2\right), && (9)\\ &= \left(l_1\cos q_1 + l_2\cos q_2\right)\mathbf{n}_1 + \left(l_1\sin q_1 + l_2\sin q_2\right)\mathbf{n}_2, && (10) \end{aligned}$$

where $\mathbf{r}^{oe}$ is the position vector of point e in the frame of reference N. Using (10), we obtain the following 2 scalar equations:

$$\begin{aligned} x &= l_1\cos q_1 + l_2\cos q_2, && (11)\\ y &= l_1\sin q_1 + l_2\sin q_2. && (12) \end{aligned}$$

Equations (11) and (12) represent the forward kinematics of the pantograph haptic device. Now we need to solve for q_2 in terms of q_1 using the holonomic constrain (1). This step is necessary since q_2 is a passive angle and has to be represented using the active angles of the device (q_1 and q_4). Using (1), we obtain

$$l_1\left(\cos q_1\mathbf{n}_1 + \sin q_1\mathbf{n}_2\right) + l_2\left(\cos q_2\mathbf{n}_1 + \sin q_2\mathbf{n}_2\right) - l_3\left(\cos q_3\mathbf{n}_1 + \sin q_3\mathbf{n}_2\right) - l_4\left(\cos q_4\mathbf{n}_1 + \sin q_4\mathbf{n}_2\right) - l_o\mathbf{n}_1 = 0, \quad (13)$$

Equation (13) provides the following two scalar equations:

$$l_1\cos q_1 + l_2\cos q_2 - l_3\cos q_3 - l_4\cos q_4 - l_o = 0, \quad (14)$$
$$l_1\sin q_1 + l_2\sin q_2 - l_3\sin q_3 - l_4\sin q_4 = 0. \quad (15)$$

Solving (14) and (15) using Newton-Raphson method. We make the following definitions:

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} l_1 \cos q_1 + l_2 \cos q_2 - l_3 \cos q_3 - l_4 \cos q_4 - l_o \\ l_1 \sin q_1 + l_2 \sin q_2 - l_3 \sin q_3 - l_4 \sin q_4 \end{pmatrix}. \tag{16}$$

All link length and q_1 and q_4 are know, therefore we solve for q_2 and q_3. The passive angles are defined as, $\mathbf{q}_\mathrm{p} = \begin{bmatrix} q_2 & q_3 \end{bmatrix}^\mathrm{T}$. Using Newton-Raphson method, $\mathbf{q}_\mathrm{p}$ is calculated using

$$\mathbf{q}_\mathrm{p}^{i+1} = \mathbf{q}_\mathrm{p}^{i} - \frac{\delta \mathbf{f}}{\delta \mathbf{q}_\mathrm{p}} \mathbf{f}. \tag{17}$$

Once q_2 is solved interns of q_1 and q_4, the configuration level kinematics is completed.

0.2 Configuration Level Inverse Kinematics

Now we solve for q_1 and q_4 based on x and y of point e. We rewrite (11) and (12) as follows:

$$x - l_1 \cos q_1 - l_2 \cos q_2 = 0, \tag{18}$$

$$y - l_1 \sin q_1 - l_2 \sin q_2 = 0. \tag{19}$$

Recall the holonomic constrain and define the following scalar equations:

$$l_1 \cos q_1 + l_2 \cos q_2 - l_3 \cos q_3 - l_4 \cos q_4 - l_o = 0, \tag{20}$$

$$l_1 \sin q_1 + l_2 \sin q_2 - l_3 \sin q_3 - l_4 \sin q_4 = 0. \tag{21}$$

Solve (18), (19), (20), and (21) using Newton-Raphson.

0.3 Motion Level Forward Kinematics

Taking the time derivative of the position vector of point e $\mathbf{r}^{oe}$ in the frame of reference N yields

$$\begin{aligned}
{}^N\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}^{oe} &= {}^N\mathbf{v}^e = \dot{x}\mathbf{n}_1 + \dot{y}\mathbf{n}_2 && (22)\\
&= {}^N\frac{\mathrm{d}}{\mathrm{d}t}\left(l_1\mathbf{a}_1 + l_2\mathbf{b}_1\right) = l_1^N\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}_1 + l_2^N\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{b}_1 && (23)\\
&= l_1\left({}^\mathrm{A}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}_1 +^\mathrm{N}\omega^\mathrm{A} \times \mathbf{a}_1\right) + l_2\left({}^\mathrm{B}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{b}_1 +^\mathrm{N}\omega^\mathrm{B} \times \mathbf{b}_1\right) && (24)\\
&= l_1\dot{q}_1\mathbf{n}_3 \times \mathbf{a}_1 + l_2\dot{q}_2\mathbf{n}_3 \times \mathbf{b}_1 && (25)\\
&= l_1\dot{q}_1\mathbf{a}_2 + l_2\dot{q}_2\mathbf{b}_2 && (26)\\
&= l_1\dot{q}_1\left(-\sin q_1\mathbf{n}_1 + \cos q_1\mathbf{n}_2\right) + l_2\dot{q}_2\left(-\sin q_2\mathbf{n}_1 + \cos q_2\mathbf{n}_2\right) && (27)
\end{aligned}$$

Finally, the forward level kinematics is given by

$$\dot{x} = -l_1 \sin q_1 \dot{q}_1 - l_2 \sin q_2 \dot{q}_2, \tag{28}$$

$$\dot{y} = l_1 \cos q_1 \dot{q}_1 + l_2 \cos q_2 \dot{q}_2. \tag{29}$$

Arranging the terms in a linear form we obtain

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin q_1 & -l_2 \sin q_2 \\ l_1 \cos q_1 & l_2 \cos q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}. \tag{30}$$

where $\mathbf{J}(\mathbf{q})$ is the Jacobian matrix of the pantograph haptic device.

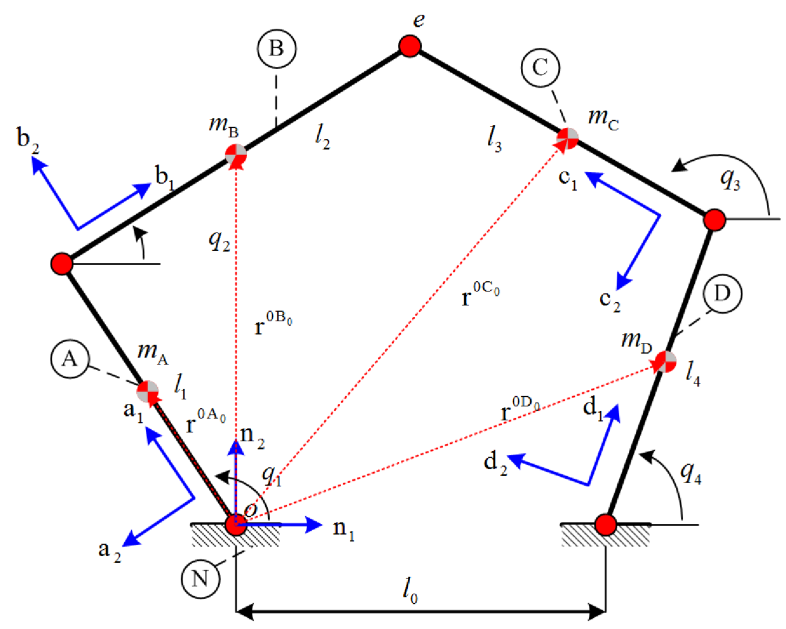

Figure 2. Dynamics of the pantograph haptic device. The red dashed vectors represent position vectors to the center of masses.

0.4 Closed-Form Solution of the Inverse Kinematics Problem

Newton-Raphson method is not recommended in implementation since it is an iterative method. Therefore, we devise the following closed-form solution of the inverse-kinematics:

$$q_1 = 2\tan^{-1}\left(\frac{-A + \sqrt{A^2 + B_1^2 - C_1^2}}{C_1 - B_1}\right) \quad \text{and} \quad q_2 = 2\tan^{-1}\left(\frac{-A - \sqrt{A^2 + B_2^2 - C_2^2}}{C_2 - B_2}\right), \tag{31}$$

where the variables A, B_1, B_2, C_1, and C_2 are given by

$$A = -2l_2 y, \tag{32}$$

$$B_1 = -2l_2\left(x - \frac{l_0}{2}\right), \tag{33}$$

$$B_2 = -2l_2\left(x + \frac{l_0}{2}\right), \tag{34}$$

$$C_1 = x^2 + y^2 + (\frac{l_0}{2})^2 + l_2^2 - l_1^2 - l_0 x, \tag{35}$$

$$C_2 = x^2 + y^2 + (\frac{l_0}{2})^2 + l_2^2 - l_1^2 + l_0 x. \tag{36}$$

1 Dynamics of the Pantograph Haptic Device

The position vector of the center of mass of link A is given by

$$\mathbf{r}^{0A_0} = \frac{l_1}{2}\mathbf{a}_1 = \frac{l_1}{2}\cos q_1 \mathbf{n}_1 + \frac{l_1}{2}\sin q_1 \mathbf{n}_2, \tag{37}$$

where $\mathbf{r}^{0\mathrm{A}_0}$ is the position vector of the center of mass of link A. The position vector of the center of mass of link B is given by

$$\begin{aligned}
\mathbf{r}^{0\mathrm{B}_0} &= l_1\mathbf{a}_1 + \frac{l_2}{2}\mathbf{b}_1, && (38)\\
&= l_1\cos q_1\mathbf{n}_1 + l_1\sin q_1\mathbf{n}_2 + \frac{l_2}{2}\cos q_2\mathbf{n}_1 + \frac{l_2}{2}\sin q_2\mathbf{n}_2, && (39)\\
&= \left(l_1\cos q_1 + \frac{l_2}{2}\cos q_2\right)\mathbf{n}_1 + \left(l_1\sin q_1 + \frac{l_2}{2}\sin q_2\right)\mathbf{n}_2. && (40)
\end{aligned}$$

Similarly, the position vector of the center of mass of link D is give by

$$\begin{aligned}
\mathbf{r}^{0\mathrm{D}_0} &= l_0\mathbf{n}_1 + \frac{l_4}{2}\mathbf{d}_1, && (41)\\
&= l_0\mathbf{n}_1 + \frac{l_4}{2}\left(\cos q_4\mathbf{n}_1 + \sin q_4\mathbf{n}_2\right), && (42)\\
&= \left(l_0 + \frac{l_4}{2}\cos q_4\right)\mathbf{n}_1 + \frac{l_4}{2}\sin q_4\mathbf{n}_2. && (43)
\end{aligned}$$

Finally, the position vector of the center of mass of link C is given by

$$\begin{aligned}
\mathbf{r}^{0\mathrm{C}_0} &= l_0\mathbf{n}_1 + l_4\mathbf{d}_1 + \frac{l_3}{2}\mathbf{c}_1, && (44)\\
&= l_0\mathbf{n}_1 + l_4\cos q_4\mathbf{n}_1 + l_4\sin q_4\mathbf{n}_2 + \frac{l_3}{2}\cos q_3\mathbf{n}_1 + \frac{l_3}{2}\sin q_3\mathbf{n}_2, && (45)\\
&= \left(l_0 + l_4\cos q_4 + \frac{l_3}{2}\cos q_3\right)\mathbf{n}_1 + \left(l_4\sin q_4 + \frac{l_3}{2}\sin q_3\right)\mathbf{n}_2. && (46)
\end{aligned}$$

Now we calculate the velocity vector of the calculated vectors in the frame of reference N as follows:

$$^{\mathrm{N}}\mathbf{v}^{\mathrm{A}_0} = {}^{\mathrm{N}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}^{0\mathrm{A}_0} = -\frac{l_1}{2}\dot{q}_1\sin q_1\mathbf{n}_1 + \frac{l_1}{2}\dot{q}_1\cos q_1\mathbf{n}_2, \tag{47}$$

where $^{\mathrm{N}}\mathbf{v}^{\mathrm{A}_0}$ is the velocity vector of the center of mass of link A in the frame of reference N. The velocity vector of the center of mass of link B is calculated using

$$^{\mathrm{N}}\mathbf{v}^{\mathrm{B}_0} = {}^{\mathrm{N}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}^{0\mathrm{B}_0} = -\left(l_1\dot{q}_1\sin q_1 + \frac{l_2}{2}\dot{q}_2\sin q_2\right)\mathbf{n}_1 + \left(l_1\dot{q}_1\cos q_1 + \frac{l_2}{2}\dot{q}_2\cos q_2\right)\mathbf{n}_2, \tag{48}$$

where $^{\mathrm{N}}\mathbf{v}^{\mathrm{B}_0}$ is the velocity vector of the center of mass of link B in the frame of reference N. Similarly, the velocity vector of the center of mass of link D is given by

$$^{\mathrm{N}}\mathbf{v}^{\mathrm{D}_0} = {}^{\mathrm{N}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}^{0\mathrm{D}_0} = -\left(\frac{l_4}{2}\dot{q}_4\sin q_4\right)\mathbf{n}_1 + \left(\frac{l_4}{2}\dot{q}_4\cos q_4\right)\mathbf{n}_2, \tag{49}$$

where $^{\mathrm{N}}\mathbf{v}^{\mathrm{D}_0}$ is the velocity vector of the center of mass of link D in the frame of reference N. Finally, the velocity vector of the center of mass of link C is calculated as follows:

$$^{\mathrm{N}}\mathbf{v}^{\mathrm{C}_0} = {}^{\mathrm{N}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}^{0\mathrm{C}_0} = -\left(l_4\dot{q}_4\sin q_4 + \frac{l_3}{2}\dot{q}_3\sin q_3\right)\mathbf{n}_1 + \left(l_4\dot{q}_4\cos q_4 + \frac{l_3}{2}\dot{q}_3\cos q_3\right)\mathbf{n}_2, \tag{50}$$

where $^{\mathrm{N}}\mathbf{v}^{\mathrm{C}_0}$ is the velocity vector of the center of mass of link C in the frame of reference N.

Link A is in rational motion only. Therefore, its kinetic energy (T_{A}) is calculated using

$$T_{\mathrm{A}} = \frac{1}{2}\frac{m_{\mathrm{A}} l_1^2}{3}\dot{q}_1^2, \tag{51}$$

Link D exhibits only rotational motion. Therefore, its kinetic energy (T_{D}) is calculated using

$$T_{\mathrm{D}} = \frac{1}{2}\frac{m_{\mathrm{D}} l_4^2}{3}\dot{q}_4^2. \tag{52}$$

The kinetic energy of Link B is due to translation and rotation and can be calculated as

$$T_{\mathrm{B}} = \frac{1}{2} I_{\mathrm{CB}} \dot{q}_2^2 + \frac{1}{2} m_{\mathrm{B}}^{\mathrm{N}} \mathbf{v}^{\mathrm{B_0}} .^{\mathrm{N}} \mathbf{v}^{\mathrm{B_0}}, \tag{53}$$

$$= \frac{1}{2}\frac{m_{\mathrm{B}} l_2^2}{12}\dot{q}_2^2 + \frac{1}{2} m_{\mathrm{B}} \left(l_1^2 \dot{q}_1^2 + \frac{l_2^2}{4}\dot{q}_2^2 + \frac{l_1 l_2}{2}\dot{q}_1 \dot{q}_2 \cos(q_1 - q_2) \right). \tag{54}$$

Similarly, the kinetic energy of Link C is due to translation and rotation and can be calculated as

$$T_{\mathrm{C}} = \frac{1}{2} I_{\mathrm{CC}} \dot{q}_3^2 + \frac{1}{2} m_{\mathrm{C}}^{\mathrm{N}} \mathbf{v}^{\mathrm{C_0}} .^{\mathrm{N}} \mathbf{v}^{\mathrm{C_0}}, \tag{55}$$

$$= \frac{1}{2}\frac{m_{\mathrm{C}} l_3^2}{12}\dot{q}_3^2 + \frac{1}{2} m_{\mathrm{C}} \left(l_4^2 \dot{q}_4^2 + \frac{l_3^2}{4}\dot{q}_3^2 + \frac{l_3 l_4}{2}\dot{q}_3 \dot{q}_4 \cos(q_3 - q_4) \right). \tag{56}$$

The total kinetic energy of the pantograph haptic device is

$$T = T_{\mathrm{A}} + T_{\mathrm{B}} + T_{\mathrm{C}} + T_{\mathrm{D}}. \tag{57}$$

Finally, the equations of motion of the pantograph haptic device are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \quad \text{for } i = 1, \ldots, 4, \tag{58}$$

where Q_i is the generalized force associated with the generalized coordinate i.

The generalized force associated with q_1 is

$$Q_1 = \mathbf{F}_{\mathrm{o}} \cdot \frac{\partial \mathbf{r}^{oe}}{\partial q_1} = -f_{\mathrm{o}x} l_1 \sin q_1 + f_{\mathrm{o}y} l_1 \cos q_1 + T_1, \tag{59}$$

where $\mathbf{F}_{\mathrm{o}}$ is the interaction force of the operator with the pantograph haptic device at a point e. Further, $f_{\mathrm{o}x}$ and $f_{\mathrm{o}y}$ are the components of $\mathbf{F}_{\mathrm{o}}$ in the frame of reference, along $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively. T_1 is the control torque input exerted on the generalized coordinate q_1. The generalized force associated with q_2 is

$$Q_2 = \mathbf{F}_{\mathrm{o}} \cdot \frac{\partial \mathbf{r}^{oe}}{\partial q_2} = -f_{\mathrm{o}x} l_2 \sin q_2 + f_{\mathrm{o}y} l_2 \cos q_2, \tag{60}$$

The generalized force associated with q_3 is

$$Q_3 = \mathbf{F}_{\mathrm{o}} \cdot \frac{\partial \mathbf{r}^{oe}}{\partial q_3} = -f_{\mathrm{o}x} l_3 \sin q_3 + f_{\mathrm{o}y} l_3 \cos q_3, \tag{61}$$

The generalized force associated with q_4 is

$$Q_4 = \mathbf{F}_{\mathrm{o}} \cdot \frac{\partial \mathbf{r}^{oe}}{\partial q_4} = -f_{\mathrm{o}x} l_4 \sin q_4 + f_{\mathrm{o}y} l_4 \cos q_4 + T_2. \tag{62}$$

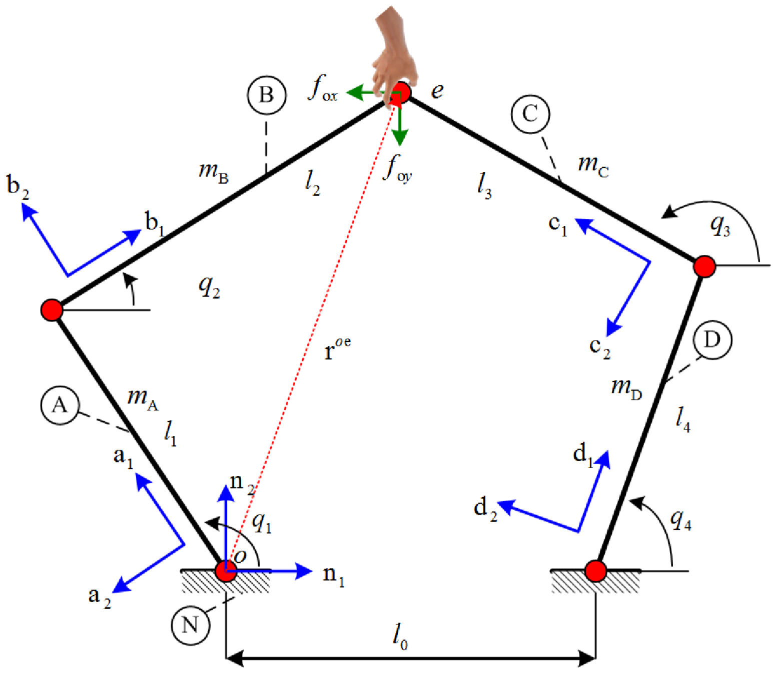

Figure 3. Generalized forces of the pantograph haptic device.

Now we recall (58) and determine the equations of motion of the pantograph haptic device as follows:

$$\left(\frac{m_A}{3} + m_B\right) l_1^2 \ddot{q}_1 + \frac{m_B l_1 l_2}{4} \ddot{q}_2 \cos(q_1 - q_2) - \frac{m_B l_1 l_2}{4} (\dot{q}_1 - \dot{q}_2) \dot{q}_2 sin(q_1 - q_2) + \frac{m_B l_1 l_2}{4} \dot{q}_1 \dot{q}_2 \sin(q_1 - q_2) = -f_{ox} l_1 \sin q_1 + f_{oy} l_1 \cos q_1 + T_1, \tag{63}$$

$$\frac{m_B l_2^2}{3} \ddot{q}_2 + \frac{m_B l_1 l_2}{4} \ddot{q}_1 \cos(q_1 - q_2) - \frac{m_B l_1 l_2}{4} (\dot{q}_1 - \dot{q}_2) \dot{q}_1 \sin(q_1 - q_2) - \frac{m_B l_1 l_2}{4} \dot{q}_1 \dot{q}_2 \sin(q_1 - q_2) = -f_{ox} l_2 \sin q_2 + f_{oy} l_2 \cos q_2, \tag{64}$$

$$\frac{m_c l_3^2}{3} \ddot{q}_3 + \frac{m_c l_3 l_4}{4} \ddot{q}_4 \cos(q_3 - q_4) - \frac{m_C l_3 l_4}{4} (\dot{q}_3 - \dot{q}_4) \dot{q}_4 \sin(q_3 - q_4) - \frac{m_C l_3 l_4}{4} \dot{q}_3 \dot{q}_4 \sin(q_3 - q_4) = -f_{ox} l_3 \sin q_3 + f_{oy} l_3 \cos q_3, \tag{65}$$

$$\left(\frac{m_D}{3} + m_C\right) l_4^2 \ddot{q}_4 + \frac{m_C l_3 l_4}{4} \ddot{q}_3 \cos(q_3 - q_4) - \frac{m_C l_3 l_4}{4} (\dot{q}_3 - \dot{q}_4) \dot{q}_3 sin(q_3 - q_4) + \frac{m_C l_3 l_4}{4} \dot{q}_3 \dot{q}_4 \sin(q_3 - q_4) = -f_{ox} l_4 \sin q_4 + f_{oy} l_4 \cos q_4 + T_2, \tag{66}$$

Based on (63), (64), (65), and (66), we define the following inertia and nonlinear damping matrices:

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} \left(\frac{m_{\mathrm{A}}}{3} + m_{\mathrm{B}}\right) l_1^2 & \frac{m_{\mathrm{B}} l_1 l_2}{4}\cos(q_1 - q_2) & 0 & 0 \\ \frac{m_{\mathrm{B}} l_1 l_2}{4}\cos(q_1 - q_2) & \frac{m_{\mathrm{B}} l_2^2}{3} & 0 & 0 \\ 0 & 0 & \frac{m_{\mathrm{c}} l_3^2}{3} & \frac{m_{\mathrm{c}} l_3 l_4}{4}\cos(q_3 - q_4) \\ 0 & 0 & \frac{m_{\mathrm{C}} l_3 l_4}{4}\cos(q_3 - q_4) & \left(\frac{m_{\mathrm{D}}}{3} + m_{\mathrm{C}}\right) l_4^2 \end{bmatrix}, \quad (67)$$

where $\mathbf{M}(\mathbf{q})$ is the inertia matrix of the pantograph haptic device. Further, the nonlinear damping matrix is given by

$$\mathbf{D}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -\frac{m_{\mathrm{B}} l_1 l_2}{4}(\dot{q}_1 - \dot{q}_2)\dot{q}_2 \sin(q_1 - q_2) + \frac{m_{\mathrm{B}} l_1 l_2}{4}\dot{q}_1\dot{q}_2 \sin(q_1 - q_2) \\ -\frac{m_{\mathrm{B}} l_1 l_2}{4}(\dot{q}_1 - \dot{q}_2)\dot{q}_1 \sin(q_1 - q_2) - \frac{m_{\mathrm{B}} l_1 l_2}{4}\dot{q}_1\dot{q}_2 \sin(q_1 - q_2) \\ -\frac{m_{\mathrm{C}} l_3 l_4}{4}(\dot{q}_3 - \dot{q}_4)\dot{q}_4 \sin(q_3 - q_4) - \frac{m_{\mathrm{C}} l_3 l_4}{4}\dot{q}_3\dot{q}_4 \sin(q_3 - q_4) \\ -\frac{m_{\mathrm{C}} l_3 l_4}{4}(\dot{q}_3 - \dot{q}_4)\dot{q}_3 \sin(q_3 - q_4) + \frac{m_{\mathrm{C}} l_3 l_4}{4}\dot{q}_3\dot{q}_4 \sin(q_3 - q_4) \end{bmatrix} \quad (68)$$

$$= \underbrace{\begin{bmatrix} \frac{m_{\mathrm{B}} l_1 l_2}{4}\dot{q}_2 s_{12} & -\frac{m_{\mathrm{B}} l_1 l_2}{4}(\dot{q}_1 - \dot{q}_2)s_{12} & 0 & 0 \\ -\frac{m_{\mathrm{B}} l_1 l_2}{4}(\dot{q}_1 - \dot{q}_2)s_{12} & -\frac{m_{\mathrm{B}} l_1 l_2}{4}\dot{q}_1 s_{12} & 0 & 0 \\ 0 & 0 & \frac{m_{\mathrm{C}} l_3 l_4}{4}\dot{q}_4 s_{34} & -\frac{m_{\mathrm{C}} l_3 l_4}{4}(\dot{q}_3 - \dot{q}_4)s_{34} \\ 0 & 0 & -\frac{m_{\mathrm{C}} l_3 l_4}{4}(\dot{q}_3 - \dot{q}_4)s_{34} & \frac{m_{\mathrm{C}} l_3 l_4}{4}\dot{q}_3 s_{34} \end{bmatrix}}_{\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} \quad (69)$$

Finally, the control torque input ($\mathbf{T}$) and interaction $\mathbf{T}_{\mathrm{h}}$ with the operator are defined as follows:

$$\mathbf{T} = \begin{bmatrix} T_1 \\ 0 \\ 0 \\ T_4 \end{bmatrix} \quad \text{and} \quad \mathbf{T}_{\mathrm{h}} = \begin{bmatrix} -f_{\mathrm{o}x} l_1 \sin q_1 + f_{\mathrm{o}y} l_1 \cos q_1 \\ -f_{\mathrm{o}x} l_2 \sin q_2 + f_{\mathrm{o}y} l_2 \cos q_2 \\ -f_{\mathrm{o}x} l_3 \sin q_3 + f_{\mathrm{o}y} l_3 \cos q_3 \\ -f_{\mathrm{o}x} l_4 \sin q_4 + f_{\mathrm{o}y} l_4 \cos q_4 \end{bmatrix}. \quad (70)$$

Now the equation of motion of the pantograph haptic device can be written in the following compact form:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{D}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{T} + \mathbf{T}_{\mathrm{h}}. \quad (71)$$

It can also be represented in the following form:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{T} + \mathbf{T}_{\mathrm{h}}. \quad (72)$$

where $\mathbf{M}(\mathbf{q})$ is the inertia matrix ($\mathbf{M}(\mathbf{q})^{\mathrm{T}} = \mathbf{M}(\mathbf{q}) > 0$). Further, $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ accounts for the centrifugal and Coriolis forces, and is related to the inertia matric via, $\left(\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\right)^{\mathrm{T}} = -\left(\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\right)$.

References

1. I. S. M. Khalil, E. Globovic, and A. Sabanovic, "High precision motion control of parallel robots with imperfections and manufacturing tolerances", in *Proceedings of the IEEE International Conference on Mechatronics (ICM)*, Istanbul, Turkey, pages 39-44, April 2011.

2. I. S. M. Khalil, V. Magdanz, S. O. Sanchez, O. G. Schmidt, and S. Misra, "Biocompatible, accurate, and fully autonomous: A sperm-driven micro-bio-robot", *Journal of Micro-Bio Robotics*, vol. 9, no. 3-4, pp. 79-86, August 2014.

3. K. Youakim, M. Ehab, O. Hatem, S. Misra, and I. S. M. Khalil, "Paramagnetic microparticles sliding on a surface: characterization and closed-loop motion control", in *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA)*, Seattle, USA, pages 4068-4073, May-June 2015.

4. I. S. M. Khalil, B. E. Wissa, B. G. Salama, and S. Stramigioli, "Wireless Motion Control of Paramagnetic Microparticles using a Magnetic-Based Robotic System with an Open-Configuration," in *Proceedings of the International Conference on Manipulation, Manufacturing and Measurement on the Nanoscale (3M Nano)*, pp. 190-196, Changchun, China, October 2015.