Robotics: Tutorial 2

Mechatronics Engineering

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Example: For the shown 2 DOF planar robot, Derive the equations for position, velocity and Acceleration level Kinematics equations (Forward and inverse):

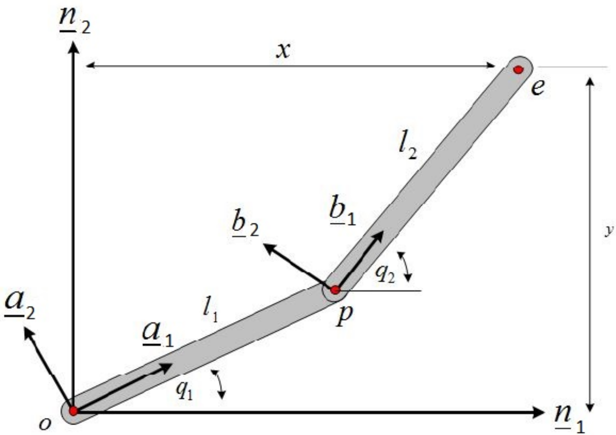

Fig. 1. Planar manipulator with 2 actuated links.

Solution: The following steps can be applied in order to reach the position-level kinematics solution:

(1) Assign a Newtonian Frame (to be used to relate everything with respect to it).For the given 2 DOF planar robot, the Newtonian frame can be assigned at the base of the robot (n_1 and n_2). Where, Newtonian frame is composed of (n_1, n_2 and n_3) and n_3 is out of the paper using right hand rule. Note that all joint angles (q) are measured with respect to the newtonian frame.

(2) Assign a local frame at each joint (preferably to align one of the axes of the frame along a link of the robot).the local frames are assigned at each joint: (a_1 and a_2) at the first joint and (b_1 and b_2) at second joint.

(3) Write Down the loop equations moving from Origin (O) to the end-effector (E) along useful paths :

- Vector from O to E.
- Moving Along robot's links.

- To go from the origin (O) to end effector (E), we can represent this as:
  $|r^{OE}$ where,
  - $|$ $\rightarrow$ vector
  - r $\rightarrow$ position
  - O $\rightarrow$ Starting point
  - E $\rightarrow$ Ending point

  which is read as position vector from O to E.

$$\boxed{|r^{OE}} = \boxed{x|n_1} + \boxed{y|n_2} \tag{1}$$

- Moving distance x along direction n_1
- Moving distance y along direction n_2

OR:

$$\boxed{|r^{OE}} = \boxed{l_1|a_1} + \boxed{l_2|b_1} \tag{2}$$

- Moving distance l_1 along direction a_1
- Moving distance l_2 along direction b_1

(4) Translate each of the frames unit vectors into the Newtonian frame, and equate both sides of the equation.

To be able to equate equation (1) and (2), they both MUST be in same coordinate frame (Newtonian frame).
So, we must convert the directions $(|a_1)$ and $(|b_1)$ to the Newtonian frame, (i.e.: this mean, we get their components along the Newtonian frame).
Therefore, get rotation matrices: describes the rotation of a certain frame with respect to another frame.

	$\|n_1$	$\|n_2$	$\|n_3$
$\|a_1$	$cos(q_1)$	$sin(q_1)$	0
$\|a_2$	$-sin(q_1)$	$cos(q_1)$	0
$\|a_3$	0	0	1

	$\|n_1$	$\|n_2$	$\|n_3$
$\|b_1$	$cos(q_2)$	$sin(q_2)$	0
$\|b_2$	$-sin(q_2)$	$cos(q_2)$	0
$\|b_3$	0	0	1

where,

$$\boxed{|a_1} = \boxed{cos(q_1)|n_1} + \boxed{sin(q_1)|n_2} \tag{3}$$

- Angle between $|a_1$ along direction $|n_1$
- projection (component) of $|a_1$ along $|n_2$ direction

Similarly,

$$\boxed{|b_1} = \boxed{cos(q_2)|n_1} + \boxed{sin(q_2)|n_2} \tag{4}$$

Substituting equations (3) and (4) in (2), and equating both sides of the equation (1) and (2):

$$|r^{OE} = l_1(cos(q_1)|n_1 + sin(q_1)|n_2) + l_2(cos(q_2)|n_1 + sin(q_2)|n_2)$$

$$x|n_1+y|n_2 = l_1(cos(q_1)|n_1+sin(q_1)|n_2)+l_2(cos(q_2)|n_1+sin(q_2)|n_2)$$

Taking common factors:

$$x|n_1 + y|n_2 = (l_1 cos(q_1) + l_2 cos(q_2))|n_1 + (l_1 sin(q_1) + l_2 sin(q_2))|n_2$$

Finally the Forward Position level Kinematics in matrix form, so that $X = f(q)$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 cos(q_1) + l_2 cos(q_2) \\ l_1 sin(q_1) + l_2 sin(q_2) \end{bmatrix}$$

Concerning the forward position level kinematics (if we know q_1 and q_2), we can calculate x and y which are the positions of the end effector.

Forward Velocity Kinematics

To find the rate of change of position along x-direction ($\dot{x}$), we take the time derivative of the position level kinematics equation:
Using Chain rule:

$$\dot{x} = \frac{dx}{dt} = \left(\frac{\partial x}{\partial q}\frac{\partial q}{\partial t}\right)$$

(once for each q, in this case q_1 and q_2).

$$\dot{x} = \underbrace{-L_1 sin(q_1)}_{\frac{\partial x}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} - \underbrace{L_2 sin(q_2)}_{\frac{\partial x}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}}, \tag{5}$$

Similarly for $\dot{y}$,

$$\dot{y} = \underbrace{L_1 cos(q_1)}_{\frac{\partial y}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} + \underbrace{L_2 cos(q_2)}_{\frac{\partial y}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}}, \tag{6}$$

Place equation (5) and (6) in a matrix form so that $\dot{X} = J(q)\dot{q}$:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 sin(q_1) & -l_2 sin(q_2) \\ l_1 cos(q_1) & l_2 cos(q_2) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad (7)$$

- $\dot{x} \rightarrow$ end effector velocities vector.
- $J(q) \rightarrow$ Jacobian Matrix.
- $\dot{q} \rightarrow$ Joints velocities vector.

Forward Acceleration level Kinematics

- since the velocity kinematics are in terms of $\dot{q}$ and $\ddot{q}$, then the chain rule will be:

$$\ddot{x} = \frac{d\dot{x}}{dt} = \left(\frac{\partial \dot{x}}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial \dot{x}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t}\right)$$

(once for each q (differentiating with respect to position) and for each $\dot{q}$ (differentiating with respect to velocity), in this case q_1 ,q_2 ,$\dot{q}_1$ and $\dot{q}_2$).

$$\ddot{x} = \underbrace{-L_1 cos(q_1)\dot{q}_1}_{\frac{\partial \dot{x}}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} \underbrace{-L_1 sin(q_1)}_{\frac{\partial \dot{x}}{\partial \dot{q}_1}} \underbrace{\ddot{q}_1}_{\frac{\partial \dot{q}_1}{\partial t}} \underbrace{-L_2 cos(q_2)\dot{q}_2}_{\frac{\partial \dot{x}}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} \underbrace{-L_2 sin(q_2)}_{\frac{\partial \dot{x}}{\partial \dot{q}_2}} \underbrace{\ddot{q}_2}_{\frac{\partial \dot{q}_2}{\partial t}}, \quad (8)$$

Similarly for $\ddot{y}$,

$$\ddot{y} = \underbrace{-L_1 sin(q_1)\dot{q}_1}_{\frac{\partial \dot{y}}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} \underbrace{+L_1 cos(q_1)}_{\frac{\partial \dot{y}}{\partial \dot{q}_1}} \underbrace{\ddot{q}_1}_{\frac{\partial \dot{q}_1}{\partial t}} \underbrace{-L_2 sin(q_2)\dot{q}_2}_{\frac{\partial \dot{y}}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} \underbrace{+L_2 cos(q_2)}_{\frac{\partial \dot{y}}{\partial \dot{q}_2}} \underbrace{\ddot{q}_2}_{\frac{\partial \dot{q}_2}{\partial t}}, \quad (9)$$

Place equation (8) and (9) in a matrix form so that
$\ddot{X} = J(q)\ddot{q} + \dot{J}(q)\dot{q}$:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} -l_1 sin(q_1) & -l_2 sin(q_2) \\ l_1 cos(q_1) & l_2 cos(q_2) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -l_1 cos(q_1)\dot{q}_1 & -l_2 cos(q_2)\dot{q}_2 \\ -l_1 sin(q_1)\dot{q}_1 & -l_2 sin(q_2)\dot{q}_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad (10)$$

- $\ddot{X} \rightarrow$ end effector acceleration components vector.
- $J(q) \rightarrow$ Jacobian Matrix.
- $\ddot{q} \rightarrow$ Joints acceleration vector.
- $\dot{J}(q) \rightarrow$ Rate of change of Jacobian Matrix.
- $\dot{q} \rightarrow$ Joints velocities vector.

Any Questions ?
Thank you