Robotics: Tutorial 3

Mechatronics Engineering

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Inverse Position Kinematics

- Introducing one of the methods available to solve inverse position kinematics level.
- Using one of the most important numerical methods, Newton Raphson.
- It depends on finding a numerical solution of a non-linear equation(s) instead of evaluating the closed form solution.

Now, For the shown non-linear function:

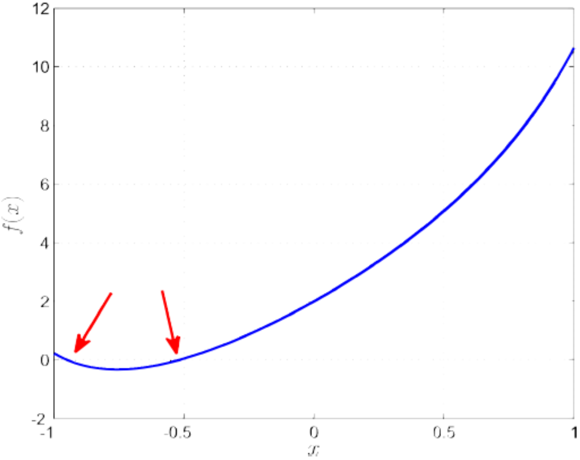


Figure: Function $f(x)$ has two roots, shown by the red arrows.

Starting from Any initial guess,we evaluate the function $F(x_0)$ at x_0.

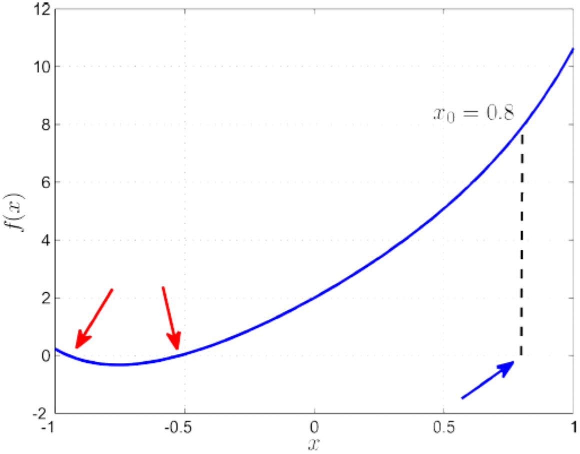


Figure: Function $f(x)$ has two roots, shown by the red arrows.

Then get the slope $\dot{F}(x_0)$, which when intersect the x-axis, produce (in most cases) a point closer to the rest of the function.

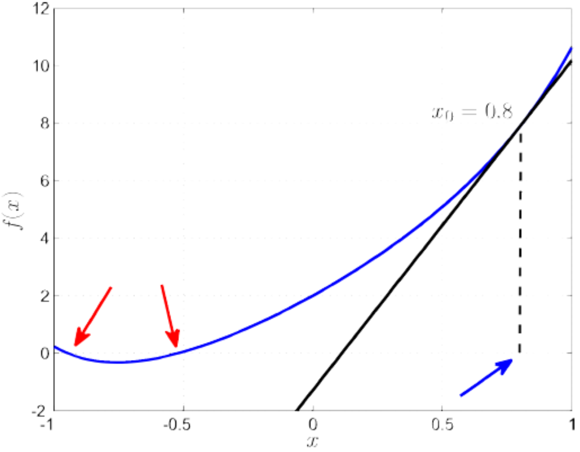


Figure: Function $f(x)$ has two roots, shown by the red arrows.

This is done in several iteration.

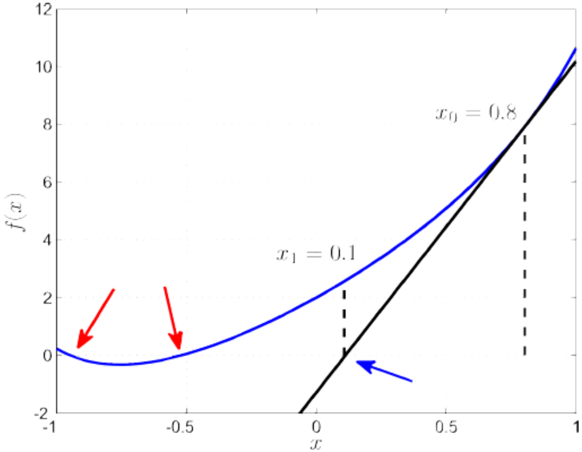


Figure: Function $f(x)$ has two roots, shown by the red arrows.

until it converge to the acceptable error value.

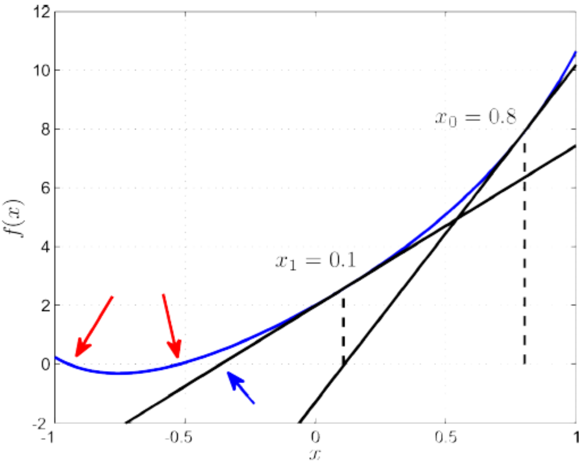


Figure: Function $f(x)$ has two roots, shown by the red arrows.

Problems of Newton Raphson

- Function diverge away from the root by choosing the wrong initial solution.
- Slope = zero at initial solution chosen.

Newton Raphson - Mathematical Approach

Let's take a look at the mathematics behind this: at the initial guess point (x_0)), the first derivative equation, which is the slope as well:

$$slope = \frac{\Delta y}{\Delta x} = \frac{F(x_0) - 0}{x_0 - x_1} \tag{1}$$

$$\dot{F}(x_0)[x_0 - x_1] = F(x_0) \tag{2}$$

$$x_0 - x_1 = \frac{F(x_0)}{\dot{F}(x_0)} \tag{3}$$

which can be written as:(this is for a single iteration, starting an initial guess).

$$x_1 = x_0 - [\dot{F}(x_0)]^{-1} F(x_0)] \tag{4}$$

In General Form:

$$x_{n+1} = x_n - [\dot{F}(x_n)]^{-1} F(x_n)] \tag{5}$$

where, n is number of current iteration.

Application of Newton Raphson in Robotics

We will expand this same formula for our application, in robotics,by changing equation from scalar into matrix and mainly to be used in the inverse position kinematics problem, where we search for the values of (q) joint angles that will make end effector reach desired goal. Thus,

$$q_{n+1} = q_n - [\frac{\partial F}{\partial q}]^{-1}|_{q=q_n} F(q_n) \tag{6}$$

Example 1

For the shown 2 DOF planar robot, Derive the inverse position level Kinematics equations:

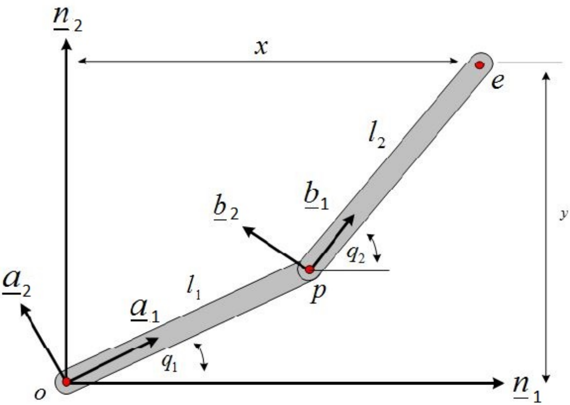


Fig. 1. Planar manipulator with 2 actuated links.

Solution

From the forward kinematics (previous tutorial):

$$x = l_1 cos(q_1) + l_2 cos(q_2) \tag{7}$$

$$l_1 cos(q_1) + l_2 cos(q_2) - x = 0 \rightarrow (f_1) \tag{8}$$

$$y = l_1 sin(q_1) + l_2 sin(q_2) \tag{9}$$

$$l_1 sin(q_1) + l_2 sin(q_2) - y = 0 \rightarrow (f_2) \tag{10}$$

Relation between joints (q) and position (x) is nonlinear.
Therefore, for the inverse kinematics solution, Newton Raphson technique is used:
We'll be searching for the values of q_1,q_2 that satisfy these function, and accordingly end-effector reach its destination:

$$\underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_1}_{\text{values after one iteration}} = \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_0}_{\text{initial values}} - \begin{bmatrix} \frac{\partial F_1}{\partial q_1} & \frac{\partial F_1}{\partial q_2} \\ \frac{\partial F_2}{\partial q_1} & \frac{\partial F_2}{\partial q_2} \end{bmatrix}^{-1} \begin{bmatrix} f_1(q_1, q_2)_0 \\ f_2(q_1, q_2)_0 \end{bmatrix} \tag{11}$$

Evaluation of different elements of equation (11) and Substitution:

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_1 = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_0 - \begin{bmatrix} -l_1 sin(q_1) & -l_2 sin(q_2) \\ l_1 cos(q_1) & l_2 cos(q_2) \end{bmatrix} \Bigg|^{-1}_{\substack{q_1=q_{10},\\ q_2=q_{20}}} \begin{bmatrix} l_1 cos(q_{10})+l_2 cos(q_{20})-x \\ l_1 sin(q_{10})+l_2 sin(q_{20})-y \end{bmatrix} \quad (12)$$

This produce values of q_1, q_2 after 1 iteration, and then this process is repeated for several iterations, in which their values converge to the acceptable values to satisfy the equations f_1, f_2 and these values become the solution of the inverse position level kinematics.

Inverse Velocity Kinematics

Relation between end effector velocities vector and Joints velocities vector is linear, and we can get q_1 and q_2 from position kinematics. The solution is:

$$\dot{q} = J^{-1}(q)\dot{X} \tag{13}$$

- $\dot{q} \rightarrow$ Joints velocities vector.
- $J^{-1}(q) \rightarrow$ Inverse of Jacobian Matrix.
  can be calculated using MATLAB.
- $\dot{x} \rightarrow$ end effector velocities vector.

Inverse Acceleration Kinematics

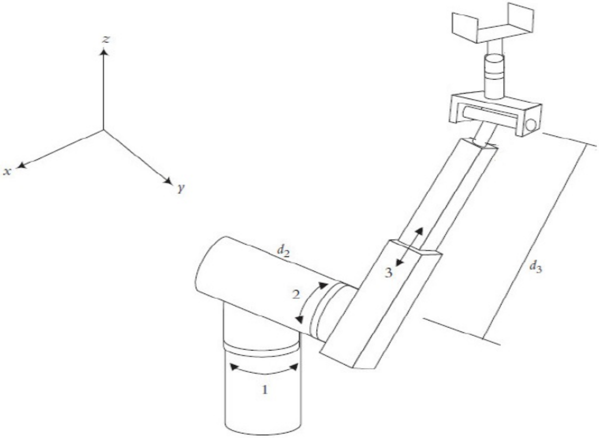
Relation between end effector acceleration vector and Joints acceleration vector is Affine (Linear relation with a shift) and we can get q_1 and q_2 from velocity kinematics.

$$\ddot{q} = J^{-1}(q)[\ddot{X} - \dot{J}(q)\dot{q}] \tag{14}$$

- $\ddot{q} \rightarrow$ Joints acceleration vector.
- $J^{-1}(q) \rightarrow$ Inverse of Jacobian Matrix.
  can be calculated using MATLAB.
- $\ddot{X} \rightarrow$ end effector acceleration vector.
- $\dot{J}(q) \rightarrow$ rate of change of Jacobian matrix.
- $\dot{q} \rightarrow$ joints velocities vector.

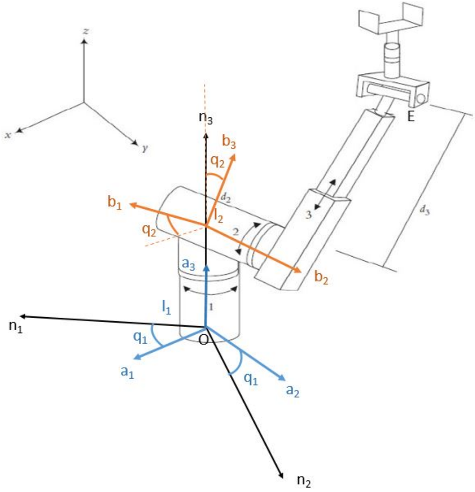
Example 2

For the shown 3 DOF non-planar robot, Derive the equations for position, velocity and acceleration level Kinematics equations (Forward and inverse):



The Forward Position Level Kinematic

First Step: Assign the frames: Newtonian Frame and local frames at each joint.



Second Step: Write Down the loop equations moving from Origin (O) to the end-effector (E):

$$|r^{OE} = x|n_1 + y|n_2 + z|n_3 \tag{15}$$

$$|r^{OE} = l_1|a_3 + l_2|b_2 + l_3|b_3 \tag{16}$$

Third Step: Derive the rotation matrices:

	$\|n_1$	$\|n_2$	$\|n_3$
$\|a_1$	$cos(q_1)$	$sin(q_1)$	0
$\|a_2$	$-sin(q_1)$	$cos(q_1)$	0
$\|a_3$	0	0	1

b_2 parallel to a_2

	$\|a_1$	$\|a_2$	$\|a_3$
$\|b_1$	$cos(q_2)$	0	$sin(q_2)$
$\|b_2$	0	1	0
$\|b_3$	$-sin(q_2)$	0	$cos(q_2)$

where,

$$|a_1 = cos(q_1)|n_1 + sin(q_1)|n_2 \tag{17}$$

$$|a_2 = -sin(q_1)|n_1 + cos(q_1)|n_2 \tag{18}$$

$$|a_3 = |n_3 \tag{19}$$

For second frame, (with respect to first frame)

$$|b_1 = cos(q_2)|a_1 + sin(q_2)|a_3 \tag{20}$$

$$|b_2 = |a_2 \tag{21}$$

$$|b_3 = -sin(q_2)|a_1 + cos(q_2)|a_3 \tag{22}$$

(with respect to Newtonian frame),From equation (18) and (21):

$$|b_2 = -sin(q_1)|n_1 + cos(q_1)|n_2 \tag{23}$$

From equation (17),(19) and (22):

$$|b_3 = -sin(q_2)cos(q_1)|n_1 - sin(q_2)sin(q_1)|n_2 + cos(q_2)|n_3 \tag{24}$$

Substituting equations (19),(23) and (24) in (15), and equating both sides of the equation (15) and (16):

$$
\begin{aligned}
|r^{OE} = {} & l_1|n_3 - l_2 sin(q_1)|n_1 + l_2 cos(q_1)|n_2 \\
& - l_3 sin(q_2)cos(q_1)|n_1 - l_3 sin(q_2)sin(q_1)|n_2 + l_3 cos(q_2)|n_3 \quad (25)
\end{aligned}
$$

$$
\begin{aligned}
x|n_1 + y|n_2 + z|n_3 = {} & l_1|n_3 - l_2 sin(q_1)|n_1 + l_2 cos(q_1)|n_2 \\
& - l_3 sin(q_2)cos(q_1)|n_1 - l_3 sin(q_2)sin(q_1)|n_2 + l_3 cos(q_2)|n_3 \quad (26)
\end{aligned}
$$

Taking common factors for the right side of the equation:

$$
\begin{aligned}
x|n_1 + y|n_2 + z|n_3 = {} & (-l_2 sin(q_1) - l_3 sin(q_2)cos(q_1))|n_1 + \\
& (l_2 cos(q_1) - l_3 sin(q_2)sin(q_1))|n_2 + (l_1 + l_3 cos(q_2))|n_3 \quad (27)
\end{aligned}
$$

In matrix form, so that $X = f(q)$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -l_2 sin(q_1) - l_3 sin(q_2) cos(q_1) \\ l_2 cos(q_1) - l_3 sin(q_2) sin(q_1) \\ l_1 + l_3 cos(q_2) \end{bmatrix} \tag{28}$$

Inverse Position Level Kinematic

Using the Newton Raphson technique:

$$\begin{bmatrix} q_1 \\ q_2 \\ l_3 \end{bmatrix}_{n+1} = \begin{bmatrix} q_1 \\ q_2 \\ l_3 \end{bmatrix}_{n} - \begin{bmatrix} \frac{\partial F_1}{\partial q_1} & \frac{\partial F_1}{\partial q_2} & \frac{\partial F_1}{\partial l_3} \\ \frac{\partial F_2}{\partial q_1} & \frac{\partial F_2}{\partial q_2} & \frac{\partial F_2}{\partial l_3} \\ \frac{\partial F_3}{\partial q_1} & \frac{\partial F_3}{\partial q_2} & \frac{\partial F_3}{\partial l_3} \end{bmatrix}^{-1}_{\substack{q_1 = q_{1n} \\ q_2 = q_{2n} \\ L_3 = L_{3n}}} \begin{bmatrix} f_1(q_1, q_2, l_3)_n \\ f_2(q_1, q_2, l_3)_n \\ f_3(q_1, q_2, l_3)_n \end{bmatrix} \tag{29}$$

and the functions are (from forward kinematics):

$-l_2 sin(q_1) - l_3 sin(q_2) cos(q_1) - x = 0 \rightarrow (f_1)$

$l_2 cos(q_1) - l_3 sin(q_2) sin(q_1) - y = 0 \rightarrow (f_2)$

$l_1 + l_3 cos(q_2) - z = 0 \rightarrow (f_3)$

By Substitution/ Evaluation of different elements of this equation:

$$\begin{bmatrix} q_1 \\ q_2 \\ l_3 \end{bmatrix}_1 = \begin{bmatrix} q_1 \\ q_2 \\ l_3 \end{bmatrix}_0 -$$

$$\left. \begin{bmatrix} -l_2 cos(q_1) + l_3 sin(q_2) sin(q_1) & -l_3 cos(q_2) cos(q_1) & -sin(q_2) cos(q_1) \\ -l_2 sin(q_1) - l_3 sin(q_2) cos(q_1) & -l_3 cos(q_2) sin(q_1) & -sin(q_2) sin(q_1) \\ 0 & -l_3 sin(q_2) & cos(q_2) \end{bmatrix} \right|^{-1}_{\substack{q_1 = q_{10} \\ q_2 = q_{20} \\ l_3 = l_{30}}}$$

$$\begin{bmatrix} -l_2 sin(q_1) - l_3 sin(q_2) cos(q_1) - x \\ l_2 cos(q_1) - l_3 sin(q_2) sin(q_1) - y \\ l_1 + l_3 cos(q_2) - z \end{bmatrix} \quad (30)$$

This produce values of q_1, q_2 and L_3 after 1 iteration, and then this process is repeated for several iterations, in which their values converge to the acceptable values to satisfy the equations f_1, f_2 and f_3 and these values become the solution of the inverse position level kinematics.

Forward Velocity Level Kinematics

Using Chain rule:

$$\frac{dx}{dt} = \left(\frac{\partial x}{\partial q}\frac{\partial q}{\partial t}\right)$$

(once for each q, in this case q_1,q_2 and L_3).

$$\begin{aligned}\dot{x} = &\underbrace{-L_2 cos(q_1)}_{\frac{\partial x}{\partial q_1}}\underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} + \underbrace{L_3 sin(q_2) sin(q_1)}_{\frac{\partial x}{\partial q_1}}\underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} - \underbrace{L_3 cos(q_2) cos(q_1)}_{\frac{\partial x}{\partial q_2}}\underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} \\ &- \underbrace{sin(q_2) cos(q_1))}_{\frac{\partial x}{\partial L_3}}\underbrace{\dot{L}_3}_{\frac{\partial L_3}{\partial t}},\end{aligned} \tag{31}$$

Similarly for $\dot{y}$,

$$\dot{y} = \underbrace{-L_2 sin(q_1)}_{\frac{\partial y}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} - \underbrace{L_3 sin(q_2) cos(q_1)}_{\frac{\partial y}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} - \underbrace{L_3 cos(q_2) sin(q_1)}_{\frac{\partial y}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}}$$
$$+ \underbrace{sin(q_2) sin(q_1)}_{\frac{\partial y}{\partial L_3}} \underbrace{\dot{L}_3}_{\frac{\partial L_3}{\partial t}}, \tag{32}$$

Similarly for $\dot{z}$,

$$\dot{z} = \underbrace{-L_3 sin(q_2)}_{\frac{\partial z}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} + \underbrace{cos(q_2)}_{\frac{\partial x}{\partial L_3}} \underbrace{\dot{L}_3}_{\frac{\partial L_3}{\partial t}}, \tag{33}$$

Place forward velocity level kinematics equation (31),(32) and (33) in a matrix form so that $\dot{X} = J(q)\dot{q}$:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -L_2 cq_1 + L_3 sq_2 sq_1 & -L_3 cq_2 cq_1 & -sq_2 cq_1 \\ -L_2 sq_1 - L_3 sq_2 cq_1 & -L_3 cq_2 sq_1 & -sq_2 sq_1 \\ 0 & -L_3 sq_2 & cq_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{L}_3 \end{bmatrix} \tag{34}$$

Inverse Velocity Level Kinematics

Place in $\dot{q} = J^{-1}(q)\dot{X}$:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{L}_3 \end{bmatrix} = \begin{bmatrix} -L_2cq_1 + L_3sq_2sq_1 & -L_3cq_2cq_1 & -sq_2cq_1 \\ -L_2sq_1 - L_3sq_2cq_1 & -L_3cq_2sq_1 & -sq_2sq_1 \\ 0 & -L_3sq_2 & cq_2 \end{bmatrix}^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \tag{35}$$

The inverse of Jacobian Matrix can be calculated using MATLAB.

Forward Acceleration Level Kinematics

Using the chain rule: $\frac{d\dot{x}}{dt} = \left(\frac{\partial \dot{x}}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial \dot{x}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t}\right)$

$$\begin{aligned}
\ddot{x} = & \underbrace{L_2 sin(q_1)\dot{q}_1}_{\frac{\partial \dot{x}}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} - \underbrace{L_2 cos(q_1)}_{\frac{\partial \dot{x}}{\partial \dot{q}_1}} \underbrace{\ddot{q}_1}_{\frac{\partial \dot{q}_1}{\partial t}} + \underbrace{L_3 cos(q_2) sin(q_1)\dot{q}_1}_{\frac{\partial \dot{x}}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} + \underbrace{L_3 sin(q_2) cos(q_1)\dot{q}_1}_{\frac{\partial \dot{x}}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} \\
& + \underbrace{L_3 sin(q_2) sin(q_1)}_{\frac{\partial \dot{x}}{\partial \dot{q}_1}} \underbrace{\ddot{q}_1}_{\frac{\partial \dot{q}_1}{\partial t}} + \underbrace{sin(q_2) sin(q_1)\dot{q}_1}_{\frac{\partial \dot{x}}{\partial l_3}} \underbrace{\dot{L}_3}_{\frac{\partial l_3}{\partial t}} + \underbrace{L_3 cos(q_2) sin(q_1)\dot{q}_2}_{\frac{\partial \dot{x}}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} \\
& + \underbrace{L_3 sin(q_2) cos(q_1)\dot{q}_2}_{\frac{\partial \dot{x}}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} - \underbrace{L_3 cos(q_2) cos(q_1)}_{\frac{\partial \dot{x}}{\partial \dot{q}_2}} \underbrace{\ddot{q}_2}_{\frac{\partial \dot{q}_2}{\partial t}} - \underbrace{cos(q_2) cos(q_1)\dot{q}_2}_{\frac{\partial \dot{x}}{\partial l_3}} \underbrace{\dot{L}_3}_{\frac{\partial l_3}{\partial t}} \\
& + \underbrace{\dot{L}_3 sin(q_2) sin(q_1)}_{\frac{\partial \dot{x}}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} - \underbrace{\dot{L}_3 cos(q_2) cos(q_1)}_{\frac{\partial \dot{x}}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} - \underbrace{sin(q_2) cos(q_1)}_{\frac{\partial \dot{x}}{\partial \dot{l}_3}} \underbrace{\ddot{L}_3}_{\frac{\partial \dot{l}_3}{\partial t}}\ ,
\end{aligned} \tag{36}$$

$$
\begin{aligned}
\ddot{y} = & -\underbrace{L_2 cos(q_1)\dot{q}_1}_{\frac{\partial \dot{y}}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} - \underbrace{L_2 sin(q_1)}_{\frac{\partial \dot{y}}{\partial \dot{q}_1}} \underbrace{\ddot{q}_1}_{\frac{\partial \dot{q}_1}{\partial t}} + \underbrace{L_3 sin(q_2) sin(q_1)\dot{q}_1}_{\frac{\partial \dot{y}}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} - \underbrace{L_3 cos(q_2) cos(q_1)\dot{q}_1}_{\frac{\partial \dot{y}}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} \\
& - \underbrace{L_3 sin(q_2) cos(q_1)}_{\frac{\partial \dot{y}}{\partial \dot{q}_1}} \underbrace{\ddot{q}_1}_{\frac{\partial \dot{q}_1}{\partial t}} - \underbrace{sin(q_2) cos(q_1)\dot{q}_1}_{\frac{\partial \dot{y}}{\partial l_3}} \underbrace{\dot{L}_3}_{\frac{\partial l_3}{\partial t}} - \underbrace{L_3 cos(q_2) cos(q_1)\dot{q}_2}_{\frac{\partial \dot{y}}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} \\
& + \underbrace{L_3 sin(q_2) sin(q_1)\dot{q}_2}_{\frac{\partial \dot{y}}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} - \underbrace{L_3 cos(q_2) sin(q_1)}_{\frac{\partial \dot{y}}{\partial \dot{q}_2}} \underbrace{\ddot{q}_2}_{\frac{\partial \dot{q}_2}{\partial t}} - \underbrace{cos(q_2) sin(q_1)\dot{q}_2}_{\frac{\partial \dot{y}}{\partial l_3}} \underbrace{\dot{L}_3}_{\frac{\partial l_3}{\partial t}} \\
& - \underbrace{\dot{L}_3 sin(q_2) cos(q_1)}_{\frac{\partial \dot{y}}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} - \underbrace{\dot{L}_3 cos(q_2) sin(q_1)}_{\frac{\partial \dot{y}}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} - \underbrace{sin(q_2) sin(q_1)}_{\frac{\partial \dot{y}}{\partial \dot{l}_3}} \underbrace{\ddot{L}_3}_{\frac{\partial \dot{l}_3}{\partial t}},
\end{aligned}
\tag{37}
$$

$$\ddot{z} = \underbrace{-L_3 cos(q_2)\dot{q}_2}_{\frac{\partial \dot{z}}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} \underbrace{-L_3 sin(q_2)}_{\frac{\partial \dot{z}}{\partial \dot{q}_2}} \underbrace{\ddot{q}_2}_{\frac{\partial \dot{q}_2}{\partial t}} \underbrace{-sin(q_2)\dot{q}_2}_{\frac{\partial \dot{z}}{\partial l_3}} \underbrace{\dot{L}_3}_{\frac{\partial l_3}{\partial t}} \underbrace{-\dot{L}_3 sin(q_2)}_{\frac{\partial \dot{z}}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} \underbrace{+cos(q_2)}_{\frac{\partial \dot{z}}{\partial \dot{l}_3}} \underbrace{\ddot{L}_3}_{\frac{\partial \dot{l}_3}{\partial t}}, \quad (38)$$

Forward Acceleration Kinematics Solution: Place equation (36),(37) and (38) in a matrix form so that $\ddot{X} = J(q)\ddot{q} + \dot{J}(q)\dot{q}$.

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -L_2 c(q_1)+L_3 s(q_2)s(q_1) & -L_3 c(q_2)c(q_1) & -s(q_2)c(q_1) \\ -L_2 s(q_1)-L_3 s(q_2)c(q_1) & -L_3 c(q_2)s(q_1) & -s(q_2)s(q_1) \\ 0 & -L_3 s(q_2) & c(q_2) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{L}_3 \end{bmatrix} +$$

$$\begin{bmatrix} L_2 sq_1 \dot{q}_1 + L_3 sq_2 cq_1 \dot{q}_1 + L_3 cq_2 sq_1 \dot{q}_2 + \dot{L}_3 sq_2 sq_1 & L_3 cq_2 sq_1 \dot{q}_1 + L_3 sq_2 cq_1 \dot{q}_2 - \dot{L}_3 cq_2 cq_1 & sq_2 sq_1 \dot{q}_1 - cq_2 cq_1 \dot{q}_2 \\ -L_2 cq_1 \dot{q}_1 + L_3 sq_2 sq_1 \dot{q}_1 - L_3 cq_2 cq_1 \dot{q}_2 - \dot{L}_3 sq_2 cq_1 & -L_3 cq_2 cq_1 \dot{q}_1 + L_3 sq_2 cq_2 \dot{q}_2 - \dot{L}_3 cq_2 sq_1 & -sq_2 cq_1 \dot{q}_1 - cq_2 sq_1 \dot{q}_2 \\ 0 & -L_3 cq_2 \dot{q}_2 - \dot{L}_3 sq_2 & -sq_2 \dot{q}_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{L}_3 \end{bmatrix} \quad (39)$$

Inverse Acceleration Level Kinematics

The inverse acceleration kinematics solution is:

$$\ddot{q} = J^{-1}(q)[\ddot{X} - \dot{J}(q)\dot{q}]$$

where, $J^{-1}(q)$ Inverse of Jacobian Matrix can be calculated using MATLAB.

Any Questions ?
Thank you