Robotics: Tutorial 4

Mechatronics Engineering

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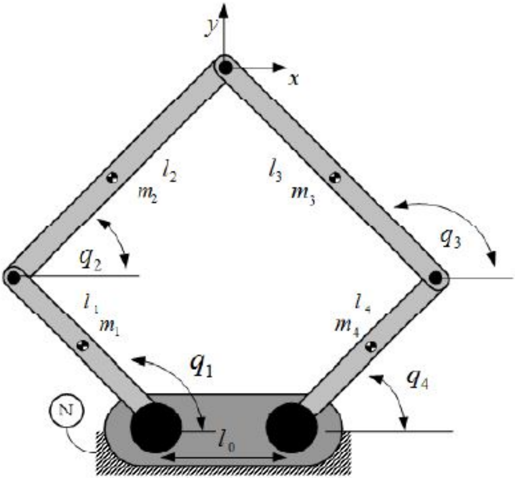
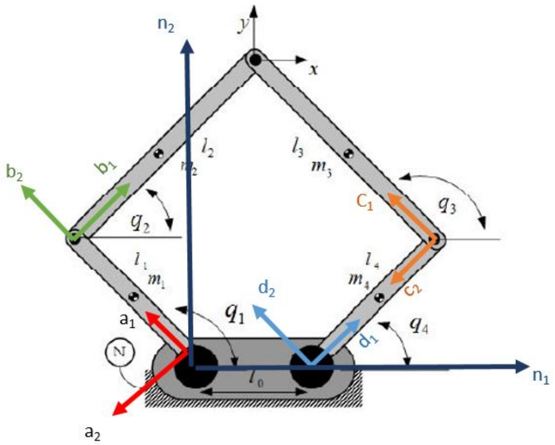
Example 1: For the shown Pantograph robot, Derive the equations for position, velocity and Acceleration level Kinematics equations (Forward and inverse):



The Forward Position Level Kinematic

First Step: Assign the frames: Newtonian Frame and local frames at each joint.



Second Step: Write Down the loop equations moving from Origin (O) to the end-effector (E):

$$|r^{OE} = x|n_1 + y|n_2 \quad (1)$$

$$|r^{OE} = l_1|a_1 + l_2|b_1 \quad (2)$$

Third Step: Derive the rotation matrices:

	$\|n_1$	$\|n_2$	$\|n_3$
$\|a_1$	$cos(q_1)$	$sin(q_1)$	0
$\|a_2$	$-sin(q_1)$	$cos(q_1)$	0
$\|a_3$	0	0	1

	$\|n_1$	$\|n_2$	$\|n_3$
$\|b_1$	$cos(q_2)$	$sin(q_2)$	0
$\|b_2$	$-sin(q_2)$	$cos(q_2)$	0
$\|b_3$	0	0	1

	n_1	n_2	n_3
c_1	$cos(q_3)$	$sin(q_3)$	0
c_2	$-sin(q_3)$	$cos(q_3)$	0
c_3	0	0	1

	n_1	n_2	n_3
d_1	$cos(q_4)$	$sin(q_4)$	0
d_2	$-sin(q_4)$	$cos(q_4)$	0
d_3	0	0	1

where,

$$|a_1 = cos(q_1)|n_1 + sin(q_1)|n_2 \tag{3}$$

For second frame,

$$|b_1 = cos(q_2)|n_1 + sin(q_2)|n_2 \tag{4}$$

For third frame,

$$|c_1 = cos(q_3)|n_1 + sin(q_3)|n_2 \tag{5}$$

For forth frame,

$$|d_1 = cos(q_4)|n_1 + sin(q_4)|n_2 \tag{6}$$

Substituting equations (3) and (4) in (2), and equating both sides of the equation (1) and (2):

$$x|n_1+y|n_2 = l_1(cos(q_1)|n_1+sin(q_1)|n_2)+l_2(cos(q_2)|n_1+sin(q_2)|n_2) \tag{7}$$

Taking common factors for the right side of the equation:

$$x|n_1+y|n_2 = (l_1 cos(q_1)+l_2 cos(q_2))|n_1+(l_1 sin(q_1)+l_2 sin(q_2))|n_2 \tag{8}$$

Since q_2 is a passive angle, it needs to be expressed in terms of active angles, using holonomic constraint equation to solve for q_2 in terms of q_1 and q_4:

$$l_1|a_1 + l_2|b_1 - l_3|c_1 - l_4|d_1 - l_0|n_1 = 0 \tag{9}$$

Substituting equations (3),(4),(5) and (6) in (9):

$$l_1(cos(q_1)|n_1 + sin(q_1)|n_2) + l_2(cos(q_2)|n_1 + sin(q_2)|n_2) - l_3(cos(q_3)|n_1 + sin(q_3)|n_2) - l_4(cos(q_4)|n_1 + sin(q_4)|n_2) - l_0|n_1 = 0 \tag{10}$$

Taking common factors:

$$(l_1 cos(q_1) + l_2 cos(q_2) - l_3 cos(q_3) - l_4 cos(q_4) - l_0|n_1 + (l_1 sin(q_1) + l_2 sin(q_2) - l_3 sin(q_3) - l_4 sin(q_4)|n_2 = 0 \tag{11}$$

$$|n_1 : l_1 cos(q_1) + l_2 cos(q_2) - l_3 cos(q_3) - l_4 cos(q_4) - l_0 = 0 \tag{12}$$

$$|n_2 : l_1 sin(q_1) + l_2 sin(q_2) - l_3 sin(q_3) - l_4 sin(q_4) = 0 \tag{13}$$

Known: $q_1, q_4, L_0, L_1, L_2, L_3$ and L_4.
Unknown: q_2, q_3.

$$\begin{bmatrix} q_2 \\ q_3 \end{bmatrix}_{n+1} = \begin{bmatrix} q_2 \\ q_3 \end{bmatrix}_n - \begin{bmatrix} \frac{\partial F_{12}}{\partial q_2} & \frac{\partial F_{12}}{\partial q_3} \\ \frac{\partial F_{13}}{\partial q_2} & \frac{\partial F_{13}}{\partial q_3} \end{bmatrix} \Bigg|^{-1}_{\substack{q_2 = q_{20} \\ q_3 = q_{30}}} \begin{bmatrix} f_{12}(q_2, q_3)_n \\ f_{13}(q_2, q_3)_n \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} q_2 \\ q_3 \end{bmatrix}_{n+1} = \begin{bmatrix} q_2 \\ q_3 \end{bmatrix}_n - \begin{bmatrix} -l_2 sin(q_2) & l_3 sin(q_3) \\ l_2 cos(q_2) & -l_3 cos(q_3) \end{bmatrix} \Bigg|^{-1}_{\substack{q_2 = q_{20} \\ q_3 = q_{30}}}$$

$$\begin{bmatrix} l_1 cos(q_1) + l_2 cos(q_{20}) - l_3 cos(q_{30}) - l_4 cos(q_4) - l_0 \\ l_1 sin(q_1) + l_2 sin(q_{20}) - l_3 sin(q_{30}) - l_4 \end{bmatrix} \tag{15}$$

Assume initial solution and after several iterations, get value for q_2 and q_3 to be substituted in the forward position level kinematics.

The Forward Position level Kinematics in matrix form, so that $X = f(q)$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 cos(q_1) + l_2 cos(q_2) \\ l_1 sin(q_1) + l_2 sin(q_2) \end{bmatrix} \tag{16}$$

The functions are (from forward kinematics):

$l_1 cos(q_1) + l_2 cos(q_2) - x = 0 \rightarrow (f_1)$

$l_1 sin(q_1) + l_2 sin(q_2) - y = 0 \rightarrow (f_2)$

$l_1 cos(q_1) + l_2 cos(q_2) - l_3 cos(q_3) - l_4 cos(q_4) - l_0 = 0 \rightarrow (f_3)$

$l_1 sin(q_1) + l_2 sin(q_2) - l_3 sin(q_3) - l_4 sin(q_4) = 0 \rightarrow (f_4)$

The Inverse Position Level Kinematic

Using the Newton Raphson technique:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}_{n+1} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}_{n} - \begin{bmatrix} \frac{\partial F_1}{\partial q_1} & \frac{\partial F_1}{\partial q_2} & \frac{\partial F_1}{\partial q_3} & \frac{\partial F_1}{\partial q_4} \\ \frac{\partial F_2}{\partial q_1} & \frac{\partial F_2}{\partial q_2} & \frac{\partial F_2}{\partial q_3} & \frac{\partial F_2}{\partial q_4} \\ \frac{\partial F_3}{\partial q_1} & \frac{\partial F_3}{\partial q_2} & \frac{\partial F_3}{\partial q_3} & \frac{\partial F_3}{\partial q_4} \\ \frac{\partial F_4}{\partial q_1} & \frac{\partial F_4}{\partial q_2} & \frac{\partial F_4}{\partial q_3} & \frac{\partial F_4}{\partial q_4} \end{bmatrix}^{-1} \Bigg|_{\substack{q_1 = q_{10} \\ q_2 = q_{20} \\ q_3 = q_{30} \\ q_4 = q_{40}}} \begin{bmatrix} f_1(q_1, q_2, q_3, q_4)_n \\ f_2(q_1, q_2, q_3, q_4)_n \\ f_3(q_1, q_2, q_3, q_4)_n \\ f_4(q_1, q_2, q_3, q_4)_n \end{bmatrix} \quad (17)$$

By Substitution/ Evaluation of different elements of this equation:

$$
\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}_1 = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}_0
$$

$$
- \begin{bmatrix} -l_1 sin(q_1) & -l_2 sin(q_2) & 0 & 0 \\ l_1 cos(q_1) & l_2 cos(q_2) & 0 & 0 \\ -l_1 sin(q_1) & -l_2 sin(q_2) & l_3 sin(q_3) & l_4 sin(q_4) \\ l_1 cos(q_1) & l_2 cos(q_2) & -l_3 cos(q_3) & -l_4 cos(q_4) \end{bmatrix}^{-1} \Bigg|_{\substack{q_1 = q_{10} \\ q_2 = q_{20} \\ q_3 = q_{30} \\ q_4 = q_{40}}}
$$

$$
\begin{bmatrix} l_1 cos(q_1) + l_2 cos(q_2) - x \\ l_1 sin(q_1) + l_2 sin(q_2) - y \\ l_1 cos(q_1) + l_2 cos(q_2) - l_3 cos(q_3) - l_4 cos(q_4) - l_0 \\ l_1 sin(q_1) + l_2 sin(q_2) - l_3 sin(q_3) - l_4 sin(q_4) \end{bmatrix} \quad (18)
$$

This produce values of q_1,q_2,q_3 and q_4 after 1 iteration, and then this process is repeated for several iterations,till we reach the solution of the inverse position level kinematics.

Forward Velocity Level Kinematics

Using Chain rule:

$$\frac{dx}{dt} = \left(\frac{\partial x}{\partial q}\frac{\partial q}{\partial t}\right)$$

(once for each q, in this case q_1,q_2,q_3 and q_4).

$$\dot{x} = \underbrace{-L_1 sin(q_1)}_{\frac{\partial x}{\partial q_1}} \ \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} \ \underbrace{-L_2 sin(q_2)}_{\frac{\partial x}{\partial q_2}} \ \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} \ , \tag{19}$$

$$\dot{y} = \underbrace{L_1 cos(q_1)}_{\frac{\partial x}{\partial q_1}} \ \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} \ + \underbrace{L_2 cos(q_2)}_{\frac{\partial x}{\partial q_2}} \ \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} \ , \tag{20}$$

$$\underbrace{-L_1 sin(q_1)}_{\frac{\partial f_3}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} \underbrace{- L_2 sin(q_2)}_{\frac{\partial f_3}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} + \underbrace{L_3 sin(q_3)}_{\frac{\partial f_3}{\partial q_3}} \underbrace{\dot{q}_3}_{\frac{\partial q_3}{\partial t}} + \underbrace{L_4 sin(q_4)}_{\frac{\partial f_3}{\partial q_4}} \underbrace{\dot{q}_4}_{\frac{\partial q_4}{\partial t}} = 0\,, \tag{21}$$

$$\underbrace{L_1 cos(q_1)}_{\frac{\partial f_4}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} + \underbrace{L_2 cos(q_2)}_{\frac{\partial f_4}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} - \underbrace{L_3 cos(q_3)}_{\frac{\partial f_4}{\partial q_3}} \underbrace{\dot{q}_3}_{\frac{\partial q_3}{\partial t}} - \underbrace{L_4 cos(q_4)}_{\frac{\partial f_4}{\partial q_4}} \underbrace{\dot{q}_4}_{\frac{\partial q_4}{\partial t}} = 0\,, \tag{22}$$

Place equation (22) in matrix form:

$$\begin{bmatrix} -l_2 sin(q_2) & l_3 sin(q_3) \\ l_2 cos(q_2) & -l_3 cos(q_3) \end{bmatrix} \begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} l_1 sin(q_1) & -l_4 sin(q_4) \\ -l_1 cos(q_1) & +l_4 cos(q_4) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_4 \end{bmatrix} \tag{23}$$

Make $\dot{q}_2$ and $\dot{q}_3$ from equation (23) the subject of the equation:

$$\begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -l_2 sin(q_2) & l_3 sin(q_3) \\ l_2 cos(q_2) & -l_3 cos(q_3) \end{bmatrix}^{-1} \begin{bmatrix} l_1 sin(q_1) & -l_4 sin(q_4) \\ -l_1 cos(q_1) & +l_4 cos(q_4) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_4 \end{bmatrix} \tag{24}$$

Using MATLAB to calculate the inverse and multiply the matrix:

$$\begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \frac{-l_1 cos(q_1)(cos(q_3) - sin(q_3))}{l_2 sin(q_2 - q_3)} & \frac{-l_4 sin(q_3 - q_4)}{l_2 sin(q_2 - q_3)} \\ \frac{-l_1 cos(q_1)(cos(q_2) - sin(q_2))}{l_3 sin(q_2 - q_3)} & \frac{-l_4 sin(q_2 - q_4)}{l_3 sin(q_2 - q_3)} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_4 \end{bmatrix} \tag{25}$$

Substitute equation (25) in equations (19) and (20):

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 sin(q_1) + \frac{l_1 cos(q_1) sin(q_2)(cos(q_3 - sin(q_3))}{sin(q_2 - q_3)} & \frac{l_4 sin(q_2) sin(q_3 - q_4)}{sin(q_2 - q_3)} \\ \frac{-l_1 cos(q_1) cos(q_3)(cos(q_2 - sin(q_2))}{sin(q_2 - q_3)} & \frac{-l_4 cos(q_2) sin(q_3 - q_4)}{sin(q_2 - q_3)} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_4 \end{bmatrix} \tag{26}$$

Now equation (26) is placed in a matrix form so that $\dot{X} = J(q)\dot{q}$ to be the solution of forward velocity level kinematics equation.

Inverse Velocity Level Kinematics

Then equation (27) is placed in a matrix form so that
$\dot{q} = J^{-1}(q)\dot{X}$ to be the solution of inverse velocity level kinematics.
The inverse of Jacobian Matrix can be calculated using MATLAB.

Forward Acceleration Level Kinematics

Using the chain rule: $\frac{d\dot{x}}{dt} = \left(\frac{\partial \dot{x}}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial \dot{x}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t} \right)$

$$\ddot{x} = \underbrace{-L_1 cos(q_1)\dot{q}_1}_{\frac{\partial \dot{x}}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} \underbrace{-L_1 sin(q_1)}_{\frac{\partial \dot{x}}{\partial \dot{q}_1}} \underbrace{\ddot{q}_1}_{\frac{\partial \dot{q}_1}{\partial t}} \underbrace{-L_2 cos(q_2)\dot{q}_2}_{\frac{\partial \dot{x}}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} \underbrace{-L_2 sin(q_2)}_{\frac{\partial \dot{x}}{\partial \dot{q}_2}} \underbrace{\ddot{q}_2}_{\frac{\partial \dot{q}_2}{\partial t}}, \quad (27)$$

$$\ddot{y} = \underbrace{-L_1 sin(q_1)\dot{q}_1}_{\frac{\partial \dot{x}}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} \underbrace{-L_1 cos(q_1)}_{\frac{\partial \dot{x}}{\partial \dot{q}_1}} \underbrace{\ddot{q}_1}_{\frac{\partial \dot{q}_1}{\partial t}} \underbrace{-L_2 sin(q_2)\dot{q}_2}_{\frac{\partial \dot{x}}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} \underbrace{+L_2 cos(q_2)}_{\frac{\partial \dot{x}}{\partial \dot{q}_2}} \underbrace{\ddot{q}_2}_{\frac{\partial \dot{q}_2}{\partial t}}, \quad (28)$$

Place equations (27) and (28) in a matrix form:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} -l_1 sin(q_1) & -l_2 sin(q_2) \\ -l_1 cos(q_1) & l_2 cos(q_2) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -l_1 cos(q_1)\dot{q}_1 & -l_2 cos(q_2)\dot{q}_2 \\ -l_1 sin(q_1)\dot{q}_1 & -l_2 sin(q_2)\dot{q}_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad (29)$$

Derive f_3 from forward velocity level kinematics equation (21):

$$\underbrace{-L_1 cos(q_1)\dot{q}_1}_{\frac{\partial \dot{f}_3}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} \underbrace{- L_1 sin(q_1)}_{\frac{\partial \dot{f}_3}{\partial \dot{q}_1}} \underbrace{\ddot{q}_1}_{\frac{\partial \dot{q}_1}{\partial t}} \underbrace{- L_2 cos(q_2)\dot{q}_2}_{\frac{\partial \dot{f}_3}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} \underbrace{- L_2 sin(q_2)}_{\frac{\partial \dot{f}_3}{\partial \dot{q}_2}} \underbrace{\ddot{q}_2}_{\frac{\partial \dot{q}_2}{\partial t}} \underbrace{+ L_3 cos(q_3)\dot{q}_3}_{\frac{\partial \dot{f}_3}{\partial q_3}} \underbrace{\dot{q}_3}_{\frac{\partial q_3}{\partial t}}$$

$$\underbrace{+ L_3 sin(q_3)}_{\frac{\partial \dot{f}_3}{\partial \dot{q}_3}} \underbrace{\ddot{q}_3}_{\frac{\partial \dot{q}_3}{\partial t}} \underbrace{+ L_4 cos(q_4)\dot{q}_4}_{\frac{\partial \dot{f}_3}{\partial q_4}} \underbrace{\dot{q}_4}_{\frac{\partial q_4}{\partial t}} \underbrace{+ L_4 sin(q_4)}_{\frac{\partial \dot{f}_3}{\partial \dot{q}_4}} \underbrace{\ddot{q}_4}_{\frac{\partial \dot{q}_4}{\partial t}} = 0, \quad (30)$$

Derive f_4 from forward velocity level kinematics equation (22):

$$\underbrace{-L_1 sin(q_1)\dot{q}_1}_{\frac{\partial \dot{f}_4}{\partial q_1}} \underbrace{\dot{q}_1}_{\frac{\partial q_1}{\partial t}} \underbrace{+L_1 cos(q_1)}_{\frac{\partial \dot{f}_4}{\partial \dot{q}_1}} \underbrace{\ddot{q}_1}_{\frac{\partial \dot{q}_1}{\partial t}} \underbrace{-L_2 sin(q_2)\dot{q}_2}_{\frac{\partial \dot{f}_4}{\partial q_2}} \underbrace{\dot{q}_2}_{\frac{\partial q_2}{\partial t}} \underbrace{+L_2 cos(q_2)}_{\frac{\partial \dot{f}_4}{\partial \dot{q}_2}} \underbrace{\ddot{q}_2}_{\frac{\partial \dot{q}_2}{\partial t}} \underbrace{+L_3 sin(q_3)\dot{q}_3}_{\frac{\partial \dot{f}_4}{\partial q_3}} \underbrace{\dot{q}_3}_{\frac{\partial q_3}{\partial t}}$$

$$\underbrace{-L_3 cos(q_3)}_{\frac{\partial \dot{f}_4}{\partial \dot{q}_3}} \underbrace{\ddot{q}_3}_{\frac{\partial \dot{q}_3}{\partial t}} \underbrace{+L_4 sin(q_4)\dot{q}_4}_{\frac{\partial \dot{f}_4}{\partial q_4}} \underbrace{\dot{q}_4}_{\frac{\partial q_4}{\partial t}} \underbrace{-L_4 cos(q_4)}_{\frac{\partial \dot{f}_4}{\partial \dot{q}_4}} \underbrace{\ddot{q}_4}_{\frac{\partial \dot{q}_4}{\partial t}} = 0, \quad (31)$$

Place equations (30) and (31) in a matrix form:

$$\begin{bmatrix} -l_2 sin(q_2) & l_3 sin(q_3) \\ l_2 cos(q_2) & -l_3 cos(q_3) \end{bmatrix} \begin{bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} -l_2 cos(q_2)\dot{q}_2 & l_3 cos(q_3)\dot{q}_3 \\ -l_2 sin(q_2)\dot{q}_2 & l_3 sin(q_3)\dot{q}_3 \end{bmatrix} \begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} l_1 sin(q_1) & -l_4 sin(q_4) \\ -l_1 cos(q_1) & l_4 cos(q_4) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_4 \end{bmatrix} + \begin{bmatrix} l_1 cos(q_1)\dot{q}_1 & -l_4 cos(q_4)\dot{q}_4 \\ l_1 sin(q_1)\dot{q}_1 & -l_4 sin(q_4)\dot{q}_4 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_4 \end{bmatrix} \quad (32)$$

Substitute equation (25) in equation (32) and rearrange equation (32) to get $\ddot{q}_2$ and $\ddot{q}_3$ in terms of $\ddot{q}_1, \ddot{q}_4, \dot{q}_1, \dot{q}_4$. Then Place this equation in a matrix form so that $\ddot{X} = J(q)\ddot{q} + \dot{J}(q)\dot{q}$ to get Forward Acceleration Kinematics Solution.

Inverse Acceleration Level Kinematics

The inverse acceleration kinematics solution is:

$$\ddot{q} = J^{-1}(q)[\ddot{X} - \dot{J}(q)\dot{q}]$$

where, $J^{-1}(q)$ Inverse of Jacobian Matrix can be calculated using MATLAB.

Any Questions ?

Thank you