Robotics: Tutorial 7

Mechatronics Engineering

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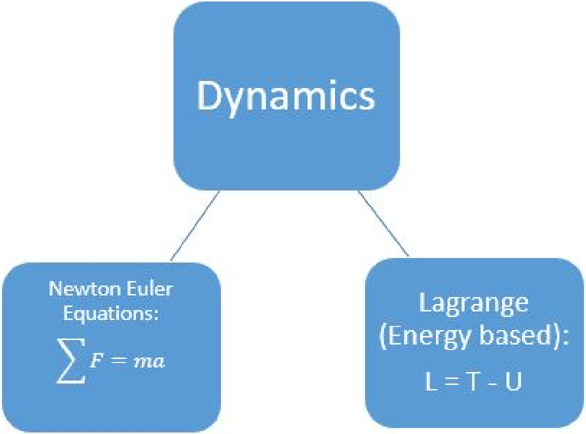
Dynamics

Used to get forces on end effector and torques at the joints, using two methods:



Lagrange method:

$$L = T - U \tag{1}$$

Thus, vector equation

$$\underbrace{T}_{\text{Total Kinetic Energy}} = \underbrace{\frac{1}{2}mV^2}_{\text{Translation}} + \underbrace{\frac{1}{2}I\omega^2}_{\text{Rotation}} \tag{2}$$

$$T = \frac{1}{2}m\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}^2 + \frac{1}{2}\underbrace{\begin{bmatrix} I_{xx} & I_{yx} & I_{zx} \\ I_{xy} & I_{yy} & I_{zy} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}_{3\times3}}_{\text{Inertia matrix}} \omega^2 \tag{3}$$

Inertia matrix includes several properties of the body, I_{xx} rotation around x only, I_{xz} rotation around 2 axes:x and z.
For link (i): (to get scalar equation because energy is scalar)

$$T_i = \frac{1}{2}m_i V_{cm_i}^T V_{cm_i} + \frac{1}{2}\omega_{1\times3}^T I_{3\times3}\omega_{3\times1} \tag{4}$$

Total Potential Energy,

$$U = mgh \tag{5}$$

For link (i):

$$U_i = m_i gh \tag{6}$$

where, g is Gravitational Vector and h is Position of cm_i.
Finally,

$$L = (T_1 + T_2 + ... + T_i) - (U_1 + U_2 + ... + U_i) \tag{7}$$

For joint (i):

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \tag{8}$$

For the shown 2 DOF robot , solve the dyanmics problem:

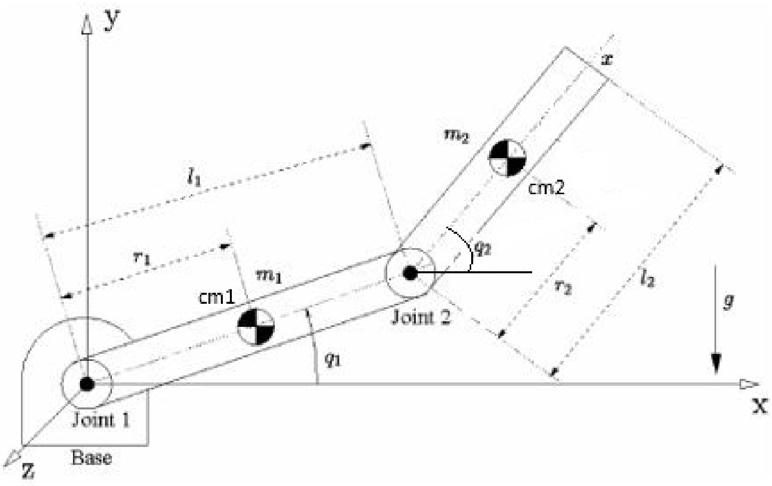


Figure: 1: 2 Link Mechanism

For link 1:
Kinetic Energy

$$|r^{ocm_1} = \begin{bmatrix} \frac{L_1}{2} cos(q_1) \\ \frac{L_1}{2} sin(q_1) \\ 0 \end{bmatrix} \tag{9}$$

$$V_{cm_1} = \begin{bmatrix} -\frac{L_1}{2} sin(q_1)\dot{q}_1 \\ \frac{L_1}{2} cos(q_1)\dot{q}_1 \\ 0 \end{bmatrix} \tag{10}$$

$$T_1 = \frac{1}{2} m_1 \begin{bmatrix} -\frac{L_1}{2} sin(q_1)\dot{q}_1 & \frac{L_1}{2} cos(q_1)\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{L_1}{2} sin(q_1)\dot{q}_1 \\ \frac{L_1}{2} cos(q_1)\dot{q}_1 \\ 0 \end{bmatrix} + \frac{1}{2} I_1 \begin{bmatrix} 0 & 0 & \dot{q}_1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \tag{11}$$

No rotation around x and y therefore equal to zero while there is rotation around z.

$$T_1 = \frac{1}{4} m_1 L_1^2 \dot{q}_1^2 sin^2(q_1) + \frac{1}{4} m_1 L_1^2 \dot{q}_1^2 cos^2(q_1) + \frac{1}{2} I_1 \dot{q}_1^2 \qquad (12)$$

Using $sin^2(q_1) + cos^2(q_1)$

$$T_1 = \frac{1}{4} m_1 L_1^2 \dot{q}_1^2 + \frac{1}{2} I_1 \dot{q}_1^2 \qquad (13)$$

For simplicity I will be written as I, no components will be derived.

$$U_1 = m_1 \begin{bmatrix} 0 & -9.81 & 0 \end{bmatrix} \begin{bmatrix} \frac{L_1}{2} cos(q_1) \\ \frac{L_1}{2} sin(q_1) \\ 0 \end{bmatrix} = -9.81 m_1 \frac{L_1}{2} sin(q_1) \qquad (14)$$

Gravitational component in this case appears only in y axis but in some cases the mechanism is oriented so it will have gravitational component in different axes.

For link 2:
Kinetic Energy

$$|r^{ocm_2} = \begin{bmatrix} L_1 cos(q_1) + \frac{L_2}{2} cos(q_2) \\ L_1 sin(q_1) + \frac{L_2}{2} sin(q_2) \\ 0 \end{bmatrix} \tag{15}$$

$$V_{cm_2} = \begin{bmatrix} -L_1 sin(q_1)\dot{q}_1 - \frac{L_2}{2} sin(q_2)\dot{q}_2 \\ L_1 cos(q_1)\dot{q}_1 + \frac{L_2}{2} cos(q_2)\dot{q}_2 \\ 0 \end{bmatrix} \tag{16}$$

$$T_2 = \frac{1}{2} m_2$$

$$\begin{bmatrix} -L_1 sin(q_1)\dot{q}_1 - \frac{L_2}{2} sin(q_2)\dot{q}_2 & L_1 cos(q_1)\dot{q}_1 + \frac{L_2}{2} cos(q_2)\dot{q}_2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -L_1 sin(q_1)\dot{q}_1 - \frac{L_2}{2} sin(q_2)\dot{q}_2 \\ L_1 cos(q_1)\dot{q}_1 + \frac{L_2}{2} cos(q_2)\dot{q}_2 \\ 0 \end{bmatrix}$$

$$+ \frac{1}{2} I_2 \begin{bmatrix} 0 & 0 & \dot{q}_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_2 \end{bmatrix} \quad (17)$$

$$T_2 = \frac{1}{2} m_2 (L_1^2 sin^2(q_1)\dot{q}_1^2 - L_1 L_2 sin(q_1)\dot{q}_1 sin(q_2)\dot{q}_2 + \frac{L_2^2}{4} sin(q_2)^2 \dot{q}_2^2$$

$$+ L_1^2 cos^2(q_1)\dot{q}1^2 + L_1 L_2 cos(q_1)\dot{q}_1 cos(q_2)\dot{q}_2 + \frac{L_2^2}{4} cos(q_2)^2 \dot{q}_2^2) + \frac{1}{2} I_2 \dot{q}_2^2 \quad (18)$$

Using $sin^2(q_1) + cos^2(q_1)$ and
$cos(q_1 + q_2) = cos(q_1)cos(q_2) - sin(q_1)sin(q_2)$

$$T_2 = \frac{1}{2} m_2 L_1^2 \dot{q}_1^2 + \frac{1}{2} m_2 L_1 L_2 cos(q_1 + q_2) \dot{q}_1 \dot{q}_2 + \frac{L_2^2}{4} m_2 \dot{q}_2^2 + \frac{1}{2} I_2 \dot{q}_2^2 \tag{19}$$

$$U_2 = m_2 \begin{bmatrix} 0 & -9.81 & 0 \end{bmatrix} \begin{bmatrix} L_1 cos(q_1) + \frac{L_2}{2} cos(q_2) \\ L_1 sin(q_1) + \frac{L_2}{2} sin(q_2) \\ 0 \end{bmatrix}$$
$$= -9.81 m_2 L_1 sin(q_1) - 9.81 m_2 \frac{L_2}{2} sin(q_2) \tag{20}$$

Substitute eqution (13), (14), (19) and (20) in:

$$L = T_1 + T_2 - U_1 - U_2 \tag{21}$$

$$L = \frac{1}{4} m_1 L_1^2 \dot{q}_1^2 + \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2} m_2 L_1^2 \dot{q}_1^2 + \frac{1}{2} m_2 L_1 L_2 cos(q_1 + q_2) \dot{q}_1 \dot{q}_2$$
$$+ \frac{L_2^2}{4} m_2 \dot{q}_2^2 + \frac{1}{2} I_2 \dot{q}_2^2 - 9.81 m_1 \frac{L_1}{2} sin(q_1) - 9.81 m_2 L_1 sin(q_1)$$
$$+ 9.81 m_2 \frac{L_2}{2} sin(q_2) \quad (22)$$

$$\tau_1 = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_1} \right) - \frac{\partial L}{\partial q_1} \quad (23)$$

$$\tau_2 = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_2} \right) - \frac{\partial L}{\partial q_2} \quad (24)$$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{1}{2} m_1 L_1^2 \dot{q}_1 + I_1 \dot{q}_1 + m_2 L_1^2 \dot{q}_1 + \frac{1}{2} m_2 L_1 L_2 cos(q_1 + q_2) \dot{q}_2 \quad (25)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_1} \right) = \frac{1}{2} m_1 L_1^2 \ddot{q}_1 + I_1 \ddot{q}_1 + m_2 L_1^2 \ddot{q}_1 + \frac{1}{2} m_2 L_1 L_2 cos(q_1 + q_2) \ddot{q}_2$$
$$+ \frac{1}{2} m_2 L_1 L_2 sin(q_1 + q_2) \dot{q}_2 (\dot{q}_1 + \dot{q}_2) \quad (26)$$

$$\frac{\partial L}{\partial q_1} = \frac{1}{2} m_2 L_1 L_2 sin(q_1+q_2)\dot{q}_1\dot{q}_2 + 9.81 m_1 \frac{L_1}{2} cos(q_1) + 9.81 m_2 L_1 cos(q_1) \quad (27)$$

$$\frac{\partial L}{\partial \dot{q}_2} = \frac{1}{2} m_2 L_1 L_2 cos(q_1 + q_2)\dot{q}_1 + \frac{L_2^2}{2} m_2 \dot{q}_2 + I_2 \dot{q}_2 \quad (28)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = \frac{1}{2} m_2 L_1 L_2 sin(q_1 + q_2)\dot{q}_1 sin(\dot{q}_1 + \dot{q}_2) + \frac{1}{2} m_2 L_1 L_2 cos(q_1 + q_2)\ddot{q}_1 + \frac{L_2^2}{2} m_2 \ddot{q}_2 + I_2 \ddot{q}_2 \quad (29)$$

$$\frac{\partial L}{\partial q_2} = \frac{1}{2} m_2 L_1 L_2 sin(q_1 + q_2)\dot{q}_1\dot{q}_2 + 9.81 m_2 \frac{L_2}{2} cos(q_2) \quad (30)$$

Substitute equations (26), (27), (29) and (30) in (23) and (24).

Any Questions ?

Thank you